MEAN VALUES OF DERIVATIVES OF L-FUNCTIONS IN FUNCTION FIELDS: I

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ABSTRACT. We investigate the first moment of the second derivative of quadratic Dirichlet L-functions over the rational function field. We establish an asymptotic formula when the cardinality of the finite field is fixed and the genus of the hyperelliptic curves associated to a family of Dirichlet L-functions over $\mathbb{F}_q(T)$ tends to infinity. As a more general result, we compute the full degree three polynomial in the asymptotic expansion of the first moment of the second derivative of this particular family of L-functions.

1. INTRODUCTION

In 1918 Hardy and Littlewood [9] proved that as $T \to \infty$

$$\int_1^T |\zeta(\tfrac{1}{2}+it)|^2 dt \sim T \log T. \tag{1.1}$$

In 1928 Ingham [11] proved a more general result by showing that as $T \to \infty$

$$\int_1^T \zeta^{(\mu)}(\tfrac{1}{2}+it)\zeta^{(\nu)}(\tfrac{1}{2}-it) \sim \frac{T}{\mu+\nu+1}(\log T)^{\mu+\nu+1}, \tag{1.2}$$

where $\zeta^{(\mu)}(s)$ denotes the μ^{th} derivative of $\zeta(s)$ and $\zeta^{(0)}(s) = \zeta(s)$. By using the simple fact that $\zeta^{(\mu)}(\frac{1}{2}-it) = \overline{\zeta^{(\mu)}(\frac{1}{2}+it)}$, it follows that

$$\int_1^T |\zeta^{(\mu)}(\tfrac{1}{2}+it)|^2 dt \sim \frac{T}{2\mu+1}(\log T)^{2\mu+1}, \tag{1.3}$$

which can be used to give (1.1) when $\mu = 0$.

The next step in the study of moments of derivatives of the Riemann zeta function was given by Gonek. In 1984 Gonek [8], in a beautiful paper,

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Date: May 18, 2016.

2010 *Mathematics Subject Classification.* Primary 11M38; Secondary 11M06, 11G20, 11M50, 14G10.

Key words and phrases. function fields, finite fields, hyperelliptic curves, derivatives of L–functions, moments of L–functions, quadratic Dirichlet L–functions, random matrix theory.

established discrete analogues of (1.1) – (1.3). If $L = \frac{1}{2\pi}\log\frac{T}{2\pi}$, one of the main results of his paper can be stated as

$$\sum_{1\le\gamma\le T}\zeta^{(\mu)}(\rho+i\alpha L^{-1})\zeta^{(\nu)}(1-\rho-i\alpha L^{-1})$$
$$= (-1)^{\mu+\nu}\left(\frac{1}{\mu+\nu+1} - H(\mu,\nu,2\pi\alpha) - H(\nu,\mu,-2\pi\alpha)\right)\frac{T}{2\pi}(\log T)^{\mu+\nu+2}$$
$$+ O(T(\log T)^{\mu+\nu+1}), \tag{1.4}$$

where $\rho = \beta + i\gamma$ denotes the non-trivial zeros of $\zeta(s)$ and

$$H(\mu,\nu,2\pi\alpha) = \mu!\sum_{l=0}^{\infty}\frac{(2\pi\alpha i)^l}{(l+\mu+1)!(l+\mu+\nu+2)}. \tag{1.5}$$

In 1988 Conrey [4] studied derivatives of the fourth moment of the Riemann zeta function. Extending and generalizing Ingham's result he proved

$$\int_1^T|\zeta'(\tfrac{1}{2}+it)|^4 \sim \frac{61}{1680\pi^2}T\left(\log\frac{T}{2\pi}\right)^8 \tag{1.6}$$

and that

$$\frac{\pi^2}{6}C_{2,m} \sim \frac{1}{16m^4}, \tag{1.7}$$

as $m\to\infty$, where

$$C_{k,m} = \lim_{T\to\infty}T^{-1}\left(\log\frac{T}{2\pi}\right)^{-k^2-2km}\int_1^T|\zeta^{(m)}(\tfrac{1}{2}+it)|^{2k}dt. \tag{1.8}$$

Recently, by studying the moments of the derivative of characteristic polynomials in $U(N)$, Conrey, Rubinstein and Snaith [5] have formulated the general conjecture that

$$\frac{1}{T}\int_0^T|\zeta'(\tfrac{1}{2}+it)|^{2k}dt \sim a_k b_k(\log T)^{k^2+2k}, \tag{1.9}$$

where the constant a_k is the same arithmetic factor that appears in the conjecture for the moments of the Riemann zeta function and is given in terms of a complicated Euler product, and the b_k is the constant coming from Random Matrix Theory, that, in this case, is given in terms of the modified Bessel function of the first kind.

The main object of this paper is to study moments of derivatives of L-functions in the function field setting. In this note, we establish the first moment of the second derivative of quadratic Dirichlet L-functions associated to hyperelliptic curves over a finite field. The main theorem of this paper can be seen as a function field analogue of Ingham's result (1.3) when $\mu = 2$ for L-functions over the rational function field.

2. Main Theorem

Before we enunciate the main theorem of this paper, we need to present some basic facts on function fields. The main reference being used for this purpose is the book by Rosen [13].

Let $\mathbb{F}_q$ be a finite field of odd cardinality $q = p^a$, with p a prime. We will denote by $A = \mathbb{F}_q[T]$ the polynomial ring over $\mathbb{F}_q$ and by $k = \mathbb{F}_q(T)$ the rational function field over $\mathbb{F}_q$.

The zeta function attached to A is defined by the following Dirichlet series,

$$\zeta_A(s) := \sum_{\substack{f \in A \\ \text{monic}}} \frac{1}{|f|^s} \qquad \text{for } \mathrm{Re}(s) > 1, \tag{2.1}$$

where $|f| = q^{\deg(f)}$ for $f \neq 0$ and $|f| = 0$ for $f = 0$. We can easily prove that

$$\zeta_A(s) = \frac{1}{1 - q^{1-s}}. \tag{2.2}$$

The quadratic Dirichlet L-function of the rational function field k is defined to be

$$L(s, \chi_D) = \sum_{\substack{f \in A \\ f \text{ monic}}} \frac{\chi_D(f)}{|f|^s} \qquad \text{for } \mathrm{Re}(s) > 1, \tag{2.3}$$

where χ_D is the quadratic character defined by the quadratic residue symbol in $\mathbb{F}_q[T]$, i.e.,

$$\chi_D(f) = \left(\frac{D}{f}\right), \tag{2.4}$$

and D is a square-free monic polynomial. In other words, if $P \in A$ is monic irreducible we have

$$\chi_D(P) = \begin{cases} 0, & \text{if } P \mid D, \\ 1, & \text{if } P \nmid D \text{ and } D \text{ is a square modulo } P, \\ -1, & \text{if } P \nmid D \text{ and } D \text{ is a non square modulo } P. \end{cases} \tag{2.5}$$

For a more detailed discussion about Dirichlet characters for function fields see [13, Chapter 3] and [3, 6].

In this paper, we work with the family of quadratic Dirichlet L-functions that are associated to polynomials $D \in \mathcal{H}_{2g+1,q}$, where

$$\mathcal{H}_{2g+1,q} = \{D \in A, \text{ square-free, monic and } \deg(D) = 2g+1\}. \tag{2.6}$$

In this case, the L–function associated to χ_D is the numerator of the zeta function associated to the hyperelliptic curve defined by the affine equation $C_D : y^2 = D(T)$ and, consequently, $L(s, \chi_D)$ is a polynomial of degree $2g$ in the variable $u = q^{-s}$ given by

$$\begin{aligned} L(s, \chi_D) &= \sum_{n=0}^{2g} A(n, \chi_D) q^{-ns} \\ &= \sum_{n=0}^{2g} \sum_{\substack{f \text{ monic} \\ \deg(f)=n}} \chi_D(f) q^{-ns}. \end{aligned} \tag{2.7}$$

(see [13, Propositions 14.6 and 17.7] and [3, Section 3]).

This L–function, as it is expected, satisfies a functional equation. Namely

$$L(s, \chi_D) = (q^{1-2s})^g L(1-s, \chi_D). \tag{2.8}$$

The Riemann hypothesis for curves, proved by Weil [14], tell us that all the zeros of $L(s, \chi_D)$ have real part equals $1/2$.

The main theorem established in this note is:

Theorem 2.1. *Let $\mathbb{F}_q$ be a fixed finite field with q odd and $D \in \mathcal{H}_{2g+1,q}$. Then*

$$
\begin{aligned}
&\sum_{D\in\mathcal{H}_{2g+1,q}} L^{''}(\tfrac{1}{2},\chi_D)\\
&=\frac{2}{3}\frac{|D|}{\zeta_A(2)}(\log(q))^2(IJ)+2\frac{|D|}{\zeta_A(2)}(\log(q))^2\left(I(J^2-K)\right)\\
&+\frac{4}{3}\frac{|D|}{\zeta_A(2)}(\log(q))^2\left(I(J^3-2KJ-J+M)\right)\\
&+\frac{2}{3}\frac{|D|}{\zeta_A(2)}(\log(q))^2 I\left(\left[\frac{g}{2}\right]+3\left[\frac{g}{2}\right]^2+2\left[\frac{g}{2}\right]^3\right)\\
&+4(\log(q))^2 I\frac{|D|}{\zeta_A(2)}g^2\left(\left[\frac{g-1}{2}\right]+1\right)+4(\log(q))^2(IJ)\frac{|D|}{\zeta_A(2)}g^2\\
&+\frac{2}{3}\frac{|D|}{\zeta_A(2)}(\log(q))^2(IJ)+2\frac{|D|}{\zeta_A(2)}(\log(q))^2\left(I(J^2-K)\right)\\
&+\frac{4}{3}\frac{|D|}{\zeta_A(2)}(\log(q))^2\left(I(J^3-2KJ-J+M)\right)\\
&+\frac{2}{3}\frac{|D|}{\zeta_A(2)}(\log(q))^2 I\left(\left[\frac{g-1}{2}\right]+3\left[\frac{g-1}{2}\right]^2+2\left[\frac{g-1}{2}\right]^3\right)\\
&+4g(\log(q))^2\left(\left[\frac{g-1}{2}\right]^2-\left[\frac{g-1}{2}\right]\right)\frac{|D|}{\zeta_A(2)}I\\
&+\left(-1+2\left[\frac{g-1}{2}\right]\right)4g(\log(q))^2\frac{|D|}{\zeta_A(2)}(-IJ)\\
&+4g(\log(q))^2\frac{|D|}{\zeta_A(2)}I(J^2-K)+O\left(|D|^{7/8}(\log_q|D|)^2\right).
\end{aligned}
\tag{2.9}
$$

Where I, J, K and M are defined as in Lemma 3.6, $[x]$ indicates the integer part of x and $|D| = q^{2g+1}$.

Using that $2g+1 = \log_q |D|$ the next result follows as a simple corollary.

Corollary 2.1. *Let $\mathbb{F}_q$ be a fixed finite field of odd cardinality q. Using the same notation as in the theorem we have,*

$$
\sum_{D\in\mathcal{H}_{2g+1,q}} L^{''}(\tfrac{1}{2},\chi_D)\sim\frac{1}{6}\frac{|D|}{\zeta_A(2)}(\log(q))^2 I(\log_q|D|)^3, \tag{2.10}
$$

as $g\to\infty$.

Remark 2.2. *The natural question to ask is why not to compute the mean value for $L^{'}(\frac{1}{2},\chi_D)$? The answer is that the mean value of $L^{'}(\frac{1}{2},\chi_D)$ is, in some sense, trivial to obtain by using the results of* [3]. *The main reason is that one has a simple formulae for $L^{'}(\frac{1}{2},\chi_D)$ in terms of $L(\frac{1}{2},\chi_D)$. Thus the mean value $L^{''}(\frac{1}{2},\chi_D)$ gives the novel information about moments of derivatives of this family of L-functions.*

Remark 2.3. *Mean values of $L^{(0)}(\frac{1}{2},\chi_D)$, i.e. first moment of quadratic L-functions over function fields, were firstly studied by Hoffstein and Rosen* [10] *and recently by Andrade and Keating* [3]*, Florea* [7] *and Jung* [12]*.*

Remark 2.4. *We will denote by $\prod_P$ products over monic and irreducible polynomials in $\mathbb{F}_q[T]$. The sums over polynomials are assumed to be sums over monic polynomials unless the contrary is stated.*

3. Preparatory Results

This section is devoted to present all the necessary preliminary results that will be used in the proof of the main theorem in the next sections. Our first preliminary result is:

Lemma 3.1 ("Approximate" Functional Equation)**.** *Let $D \in \mathcal{H}_{2g+1,q}$. Then $L(s,\chi_D)$ can be represented as*

$$L(s,\chi_D) = \sum_{\substack{f_1 \text{ monic} \\ \deg(f_1)\leq g}} \frac{\chi_D(f_1)}{|f_1|^s} + (q^{1-2s})^g \sum_{\substack{f_2 \text{ monic} \\ \deg(f_2)\leq g-1}} \frac{\chi_D(f_2)}{|f_2|^{1-s}}. \tag{3.1}$$

Proof. The proof of this Lemma is to be found in [3, Lemma 3.3]. □

The next lemma is taken from Andrade–Keating [3, Proposition 5.2] and it is about counting the number of square–free polynomials coprime to a fixed monic polynomial.

Lemma 3.2. *Let $f \in A$ be a fixed monic polynomial. Then for all $\varepsilon > 0$ we have that*

$$\sum_{\substack{D\in\mathcal{H}_{2g+1,q} \\ (D,f)=1}} 1 = \frac{|D|}{\zeta_A(2)} \prod_{\substack{P \text{ monic} \\ \text{irreducible} \\ P|f}} \left(\frac{|P|}{|P|+1}\right) + O\left(|D|^{\frac{1}{2}}|f|^{\varepsilon}\right). \tag{3.2}$$

Our next lemma is that

Lemma 3.3.

$$\sum_{\substack{l \text{ monic} \\ \deg(l)=n}} \prod_{P|l} (1+|P|^{-1})^{-1} = q^n \sum_{\substack{d \text{ monic} \\ \deg(d)\leq n}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1}. \tag{3.3}$$

Proof. This is Lemma 5.7 in [3]. □

We now can prove our next auxiliary result.

Lemma 3.4. *Let $l \geq 0$ be an integer. Then we have*

$$\sum_{\substack{d \text{ monic} \\ \deg(d)>[g/2]}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} (deg(d))^l = O\left(g^l q^{-g/2}\right). \tag{3.4}$$

Proof.

$$
\begin{aligned}
&\sum_{\substack{d \text{ monic} \\ \deg(d)>[g/2]}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} (\deg(d))^l \\
&\leq \sum_{\substack{d \text{ monic} \\ \deg(d)>[g/2]}} \frac{\mu^2(d)}{|d|} \prod_{P|d} \frac{1}{|P|} (\deg(d))^l \\
&= \sum_{h>[g/2]} \sum_{\substack{d \text{ monic} \\ \deg(d)=h}} h^l q^{-2h} \qquad (3.5) \\
&= \sum_{h>[g/2]} h^l q^{-h} \\
&\ll g^l q^{-g/2}.
\end{aligned}
$$

□

Lemma 3.5. *With the same notation as before, we have*

$$
\sum_{d \text{ monic}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} = \prod_P \left(1 - \frac{1}{|P|(|P|+1)}\right). \tag{3.6}
$$

Proof. This is Lemma 5.9 in [3]. □

The next lemma can be seen as a generalization of the previous lemma.

Lemma 3.6. *Let*

$$
I = \prod_P \left(1 - \frac{1}{|P|(|P|+1)}\right), \tag{3.7}
$$

$$
J = \sum_{\substack{P \text{ monic} \\ \text{irreducible}}} \frac{\deg(P)}{|P|(|P|+1)-1}, \tag{3.8}
$$

$$
K = \sum_{\substack{P \text{ monic} \\ \text{irreducible}}} \frac{(\deg(P))^2(|P|(|P|+1))}{(|P|(|P|+1)-1)^2}, \tag{3.9}
$$

and

$$
M = \sum_{\substack{P \text{ monic} \\ \text{irreducible}}} \frac{(\deg(P))^3(|P|(|P|+1)(|P|(|P|+1)+1))}{(|P|(|P|+1)-1)^3}. \tag{3.10}
$$

Then we have that,

(i)

$$
\sum_{d \text{ monic}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} \deg(d) = -IJ. \tag{3.11}
$$

(ii)

$$\sum_{d \text{ monic}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} (\deg(d))^2 = I(J^2 - K). \tag{3.12}$$

(iii)

$$\sum_{d \text{ monic}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} (\deg(d))^3 = -I(J^3 - 2KJ - J + M). \tag{3.13}$$

Proof. To prove (i), let,

$$f(s) = \sum_{d \text{ monic}} \deg(d) \frac{\mu(d)}{|d|^s} \prod_{P|d} \frac{1}{|P|+1} \tag{3.14}$$

and

$$g(s) = \sum_{d \text{ monic}} \frac{\mu(d)}{|d|^s} \prod_{P|d} \frac{1}{|P|+1}. \tag{3.15}$$

A simple calculation shows that

$$g'(s) = -f(s) \log q \tag{3.16}$$

and by a variant of Lemma 3.5

$$g(s) = \prod_{P} \left(1 - \frac{1}{|P|^s(|P|+1)}\right). \tag{3.17}$$

Computing $g'(s)$ using (3.17) and the product rule gives us

$$g'(s) = g(s) \log q \sum_{\substack{P \text{ monic} \\ \text{irreducible}}} \frac{\deg(P)}{|P|^s(|P|+1) - 1}. \tag{3.18}$$

Combining (3.16) and (3.18) we have that

$$f(s) = -g(s) \sum_{\substack{P \text{ monic} \\ \text{irreducible}}} \frac{\deg(P)}{|P|^s(|P|+1) - 1}. \tag{3.19}$$

Putting $s = 1$ proves (i).

Mimicking the proof of (i), by considering $f^{'}(s)$ and $f^{''}(s)$ respectively, one can prove (ii) and (iii) and so obtain the full lemma. □

The last result of this section is about non-trivial sums involving quadratic characters. It can be proved by using the Riemann Hypothesis for curves and the Pólya-Vinogradov inequality for function fields.

Proposition 3.7. *If $f \in \mathbb{F}_q[x]$ is not a square then*

$$\Big| \sum_{D \in \mathcal{H}_{2g+1,q}} \chi_D(f) \Big| \ll |D|^{1/2} |f|^{1/4}. \tag{3.20}$$

(For a proof see [1, 2]*).*

4. Proof of the Main Theorem

By taking the second derivative of the approximate functional equation (3.1) for $L(s, \chi_D)$ we obtain that

$$\begin{aligned} L''(s, \chi_D) &= (\log(q))^2 \sum_{n=0}^{g} n^2 q^{-ns} A(n, \chi_D) \\ &\quad + 4g^2 (\log(q))^2 (q^{1-2s})^g \sum_{m=0}^{g-1} A(m, \chi_D) q^{m(s-1)} \\ &\quad + (\log(q))^2 (q^{1-2s})^g \sum_{m=0}^{g-1} m^2 q^{m(1-s)} A(m, \chi_D) \\ &\quad - 4g (\log(q))^2 (q^{1-2s})^g \sum_{m=0}^{g-1} m q^{m(1-s)} A(m, \chi_D), \end{aligned} \tag{4.1}$$

where

$$A(n, \chi_D) = \sum_{\substack{f \text{ monic} \\ \deg(f) = n}} \chi_D(f). \tag{4.2}$$

For $s = \frac{1}{2}$, the equation above simplifies to

$$\begin{aligned} L''(\tfrac{1}{2}, \chi_D) &= (\log(q))^2 \sum_{n=0}^{g} n^2 q^{-n/2} A(n, \chi_D) \\ &\quad + 4g^2 (\log(q))^2 \sum_{m=0}^{g-1} A(m, \chi_D) q^{-m/2} \\ &\quad + (\log(q))^2 \sum_{m=0}^{g-1} m^2 q^{-m/2} A(m, \chi_D) \\ &\quad - 4g (\log(q))^2 \sum_{m=0}^{g-1} m q^{-m/2} A(m, \chi_D) \\ &= S_1 + S_2 + S_3 + S_4. \end{aligned} \tag{4.3}$$

The task now is to average (4.3) over $\mathcal{H}_{2g+1,q}$. We will accomplish this task for each of the sums S_i, $i = 1, \ldots, 4$ in (4.3).

4.1. **Averaging S_1.** The main result in this section is encapsulated in the following proposition.

Proposition 4.1. *Using the same notation from Lemma 3.6, we have that*

$$\begin{aligned}
&\sum_{D\in\mathcal{H}_{2g+1,q}} (\log(q))^2 \sum_{n=0}^{g} n^2 q^{-n/2} A(n,\chi_D) \\
&= \frac{2}{3}\frac{|D|}{\zeta_A(2)}(\log(q))^2(IJ) + 2\frac{|D|}{\zeta_A(2)}(\log(q))^2\left(I(J^2-K)\right) \\
&+ \frac{4}{3}\frac{|D|}{\zeta_A(2)}(\log(q))^2\left(I(J^3-2KJ-J+M)\right) \\
&+ \frac{2}{3}\frac{|D|}{\zeta_A(2)}(\log(q))^2 I\left(\left[\frac{g}{2}\right]+3\left[\frac{g}{2}\right]^2+2\left[\frac{g}{2}\right]^3\right) + O\left(|D|^{7/8}(\log_q|D|)^2\right).
\end{aligned} \tag{4.4}$$

We start by splitting the summation according f being a perfect square of a polynomial in $\mathbb{F}_q[T]$ and for those f that are not a perfect square. In other words, we can write the average over S_1 as

$$\begin{aligned}
&\sum_{D\in\mathcal{H}_{2g+1,q}} (\log(q))^2 \sum_{n=0}^{g} n^2 q^{-n/2} A(n,\chi_D) \\
&= \sum_{D\in\mathcal{H}_{2g+1,q}} (\log(q))^2 \sum_{n=0}^{g} n^2 q^{-n/2} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f=\square}} \chi_D(f) \\
&+ \sum_{D\in\mathcal{H}_{2g+1,q}} (\log(q))^2 \sum_{n=0}^{g} n^2 q^{-n/2} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f\neq\square}} \chi_D(f).
\end{aligned} \tag{4.5}$$

For the non-square contribution (i.e. $f \neq \square$) we can prove the following lemma.

Lemma 4.2. *If $f \in \mathbb{F}_q[T]$ is a non-square polynomial then we have*

$$\sum_{D\in\mathcal{H}_{2g+1,q}} (\log(q))^2 \sum_{n=0}^{g} n^2 q^{-n/2} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f\neq\square}} \chi_D(f) = O\left(|D|^{7/8}(\log_q|D|)^2\right). \tag{4.6}$$

Proof. We have that

(4.7)
$$\Big|\sum_{D\in\mathcal{H}_{2g+1,q}}(\log(q))^2\sum_{n=0}^{g}n^2q^{-n/2}\sum_{\substack{f \text{ monic}\\ \deg(f)=n\\ f\neq\square}}\chi_D(f)\Big| \ll \Big|\sum_{n=0}^{g}n^2q^{-n/2}\sum_{\substack{f \text{ monic}\\ \deg(f)=n\\ f\neq\square}}\sum_{D\in\mathcal{H}_{2g+1,q}}\chi_D(f)\Big|.$$

We now use Proposition 3.7 to obtain that

$$\Big|\sum_{n=0}^{g}n^2q^{-n/2}\sum_{\substack{f \text{ monic}\\ \deg(f)=n\\ f\neq\square}}\sum_{D\in\mathcal{H}_{2g+1,q}}\chi_D(f)\Big| \ll \sum_{n=0}^{g}n^2q^{-n/2}\sum_{\substack{f \text{ monic}\\ \deg(f)=n}}|D|^{1/2}|f|^{1/4}$$
$$\ll q^{\frac{7}{4}g}g^2, \tag{4.8}$$

and this concludes the proof of the lemma. □

For the f a perfect square of a polynomial in $\mathbb{F}_q[T]$, the calculations are more involving and laborius. Our main result is given in the next proposition.

Proposition 4.3. *Using the same notation as before, we conclude that*

$$\sum_{D\in\mathcal{H}_{2g+1,q}}(\log(q))^2\sum_{n=0}^{g}n^2q^{-n/2}\sum_{\substack{f \text{ monic}\\ \deg(f)=n\\ f=\square}}\chi_D(f)$$
$$= \frac{2}{3}\frac{|D|}{\zeta_A(2)}(\log(q))^2(IJ) + 2\frac{|D|}{\zeta_A(2)}(\log(q))^2\left(I(J^2-K)\right)$$
$$+ \frac{4}{3}\frac{|D|}{\zeta_A(2)}(\log(q))^2\left(I(J^3-2KJ-J+M)\right)$$
$$+ \frac{2}{3}\frac{|D|}{\zeta_A(2)}(\log(q))^2 I\left(\left[\frac{g}{2}\right]+3\left[\frac{g}{2}\right]^2+2\left[\frac{g}{2}\right]^3\right)+O\left(|D|^{3/4}(\log_q|D|)^3\right). \tag{4.9}$$

We can write

$$
\begin{aligned}
&\sum_{D\in\mathcal{H}_{2g+1,q}} (\log(q))^2 \sum_{n=0}^{g} n^2 q^{-n/2} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f=\square}} \chi_D(f) \\
&= (\log(q))^2 \sum_{n=0}^{g} n^2 q^{-n/2} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f=l^2}} \sum_{D\in\mathcal{H}_{2g+1,q}} \chi_D(l^2). \qquad (4.10)
\end{aligned}
$$

By an application of Lemma 3.2 and Lemma 3.3, and after some arithmetic manipulations we obtain that

$$
\begin{aligned}
&\sum_{D\in\mathcal{H}_{2g+1,q}} (\log(q))^2 \sum_{n=0}^{g} n^2 q^{-n/2} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f=\square}} \chi_D(f) \\
&= \frac{|D|}{\zeta_A(2)} (\log(q))^2 \sum_{\substack{n=0 \\ 2|n}}^{g} n^2 \sum_{\substack{d \text{ monic} \\ \deg(d)\leq n/2}} \frac{\mu(d)}{|d|} \prod_{P|d} \left(\frac{1}{|P|+1}\right) + O\left(|D|^{1/2} \sum_{n=0}^{g} n^2 q^{n\varepsilon}\right) \\
&= \frac{4|D|}{\zeta_A(2)} (\log(q))^2 \sum_{\substack{d \text{ monic} \\ \deg(d)\leq [g/2]}} \frac{\mu(d)}{|d|} \prod_{P|d} \left(\frac{1}{|P|+1}\right) \sum_{\deg(d)\leq m\leq [g/2]} m^2 \\
&+ O\left(|D|^{1/2+\varepsilon} (\log_q |D|)^2\right). \qquad (4.11)
\end{aligned}
$$

By performing the sum over m in the previous equation, we obtain

$$\sum_{D\in\mathcal{H}_{2g+1,q}}(\log(q))^2\sum_{n=0}^{g}n^2q^{-n/2}\sum_{\substack{f \text{ monic}\\ \deg(f)=n\\ f=\square}}\chi_D(f)$$
$$=\frac{4|D|}{\zeta_A(2)}(\log(q))^2\sum_{\substack{d \text{ monic}\\ \deg(d)\leq[g/2]}}\frac{\mu(d)}{|d|}\prod_{P|d}\left(\frac{1}{|P|+1}\right)\left(\frac{1}{6}\left(\deg(d)(-1+3\deg(d)-2(\deg(d))^2)\right)\right)$$
$$+\frac{4|D|}{\zeta_A(2)}(\log(q))^2\sum_{\substack{d \text{ monic}\\ \deg(d)\leq[g/2]}}\frac{\mu(d)}{|d|}\prod_{P|d}\left(\frac{1}{|P|+1}\right)\left(\frac{1}{6}\left(\left[\frac{g}{2}\right]+3\left[\frac{g}{2}\right]^2+2\left[\frac{g}{2}\right]^3\right)\right)$$
$$+O\left(|D|^{1/2+\varepsilon}(\log_q|D|)^2\right). \tag{4.12}$$

We will establish four lemmas that will be used to prove the Proposition 4.3.

Lemma 4.4. *We have that,*

$$\frac{4|D|}{\zeta_A(2)}(\log(q))^2\sum_{\substack{d \text{ monic}\\ \deg(d)\leq[g/2]}}\frac{\mu(d)}{|d|}\prod_{P|d}\left(\frac{1}{|P|+1}\right)\left(\frac{1}{6}\left(\left[\frac{g}{2}\right]+3\left[\frac{g}{2}\right]^2+2\left[\frac{g}{2}\right]^3\right)\right)$$
$$=\frac{2}{3}\frac{|D|}{\zeta_A(2)}(\log(q))^2 I\left(\left[\frac{g}{2}\right]+3\left[\frac{g}{2}\right]^2+2\left[\frac{g}{2}\right]^3\right)+O\left(|D|^{3/4}(\log_q|D|)^3\right). \tag{4.13}$$

Proof. We write the sum over d as

$$\sum_{\substack{d \text{ monic}\\ \deg(d)\leq[g/2]}}=\sum_{d \text{ monic}}-\sum_{\substack{d \text{ monic}\\ \deg(d)>[g/2]}}. \tag{4.14}$$

For the sum over all monic d we use Lemma 3.5 and for the sum over $\deg(d) > [g/2]$ we use Lemma 3.4. This completes the proof of the lemma. ☐

Our next result is given below.

Lemma 4.5. *Using the same notation as in Lemma 3.6, we have that*

$$\frac{4}{6}\frac{|D|}{\zeta_A(2)}(\log(q))^2\sum_{\substack{d \text{ monic}\\ \deg(d)\leq[g/2]}}\frac{\mu(d)}{|d|}\prod_{P|d}\left(\frac{1}{|P|+1}\right)(-\deg(d))$$
$$=\frac{2}{3}\frac{|D|}{\zeta_A(2)}(\log(q))^2(IJ)+O\left(|D|^{3/4}(\log_q|D|)\right). \tag{4.15}$$

Proof. We write the sum over d as

$$\sum_{\substack{d \text{ monic} \\ \deg(d)\leq[g/2]}} = \sum_{d \text{ monic}} - \sum_{\substack{d \text{ monic} \\ \deg(d)>[g/2]}} . \tag{4.16}$$

For the sum over all monic d we use part (i) from Lemma 3.6 and for the sum over $\deg(d) > [g/2]$ we use Lemma 3.4. This completes the proof of the lemma. □

Lemma 4.6. *Using the same notation as in Lemma 3.6, we find that*

$$\frac{12}{6}\frac{|D|}{\zeta_A(2)}(\log(q))^2 \sum_{\substack{d \text{ monic} \\ \deg(d)\leq[g/2]}} \frac{\mu(d)}{|d|}\prod_{P|d}\left(\frac{1}{|P|+1}\right)(\deg(d))^2$$

$$= 2\frac{|D|}{\zeta_A(2)}(\log(q))^2\left(I(J^2-K)\right) + O\left(|D|^{3/4}(\log_q|D|)^2\right). \tag{4.17}$$

Proof. We write the sum over d as

$$\sum_{\substack{d \text{ monic} \\ \deg(d)\leq[g/2]}} = \sum_{d \text{ monic}} - \sum_{\substack{d \text{ monic} \\ \deg(d)>[g/2]}} . \tag{4.18}$$

For the sum over all monic d we use part (ii) from Lemma 3.6 and for the sum over $\deg(d) > [g/2]$ we use Lemma 3.4. This completes the proof of the lemma. □

And our last lemma is

Lemma 4.7. *We have that,*

$$-\frac{8}{6}\frac{|D|}{\zeta_A(2)}(\log(q))^2 \sum_{\substack{d \text{ monic} \\ \deg(d)\leq[g/2]}} \frac{\mu(d)}{|d|}\prod_{P|d}\left(\frac{1}{|P|+1}\right)(\deg(d))^3$$

$$= \frac{4}{3}\frac{|D|}{\zeta_A(2)}(\log(q))^2\left(I(J^3-2KJ-J+M)\right) + O\left(|D|^{3/4}(\log_q|D|)^3\right). \tag{4.19}$$

Proof. We write the sum over d as

$$\sum_{\substack{d \text{ monic} \\ \deg(d)\leq[g/2]}} = \sum_{d \text{ monic}} - \sum_{\substack{d \text{ monic} \\ \deg(d)>[g/2]}} . \tag{4.20}$$

For the sum over all monic d we use part (iii) from Lemma 3.6 and for the sum over $\deg(d) > [g/2]$ we use Lemma 3.4. This completes the proof of the lemma. □

Proof of Proposition 4.3. By using equation (4.12) and Lemma 4.4, Lemma 4.5, Lemma 4.6 and Lemma 4.7 we establish the desired proposition. □

Finally we can prove the main result in this section.

Proof of Proposition 4.1. Using Lemma 4.2 and Proposition 4.3 in equation (4.5) finishes the proof of this proposition. □

4.2. **Averaging S_2.** The main result in this section is given below.

Proposition 4.8. *Using the same notation as Proposition 4.1, we can deduce that*

$$\begin{aligned}&\sum_{D\in\mathcal{H}_{2g+1,q}} 4g^2(\log(q))^2\sum_{m=0}^{g-1} A(m,\chi_D)q^{-m/2}\\ &= 4(\log(q))^2 I\frac{|D|}{\zeta_A(2)}g^2\left(\left[\frac{g-1}{2}\right]+1\right)+4(\log(q))^2(IJ)\frac{|D|}{\zeta_A(2)}g^2\\ &+O\left(|D|^{7/8}(\log_q|D|)^2\right).\end{aligned}\tag{4.21}$$

Proof. It has been shown in [3, Equation (7.2)] that

$$\begin{aligned}&\sum_{D\in\mathcal{H}_{2g+1,q}}\sum_{m=0}^{g-1} A(m,\chi_D)q^{-m/2}\\ &= I\frac{|D|}{\zeta_A(2)}\left(\left[\frac{g-1}{2}\right]+1\right)+(IJ)\frac{|D|}{\zeta_A(2)}+O(2^g q^{\frac{3}{2}g+\frac{3}{4}}).\end{aligned}\tag{4.22}$$

By making use of Proposition 3.7 we can change the error term of the above equation (4.22) to be $O\left(|D|^{7/8}\right)$. Combinining this new error term with the main term in the equation (4.22) we establish the desired result. □

4.3. **Averaging S_3.** The main result is given by

Proposition 4.9. *Using the same notation as Proposition 4.1, we have that*

$$\begin{aligned}&\sum_{D\in\mathcal{H}_{2g+1,q}} (\log(q))^2\sum_{m=0}^{g-1} m^2q^{-m/2}A(m,\chi_D)\\ &=\frac{2}{3}\frac{|D|}{\zeta_A(2)}(\log(q))^2(IJ)+2\frac{|D|}{\zeta_A(2)}(\log(q))^2\left(I(J^2-K)\right)\\ &+\frac{4}{3}\frac{|D|}{\zeta_A(2)}(\log(q))^2\left(I(J^3-2KJ-J+M)\right)\\ &+\frac{2}{3}\frac{|D|}{\zeta_A(2)}(\log(q))^2 I\left(\left[\frac{g-1}{2}\right]+3\left[\frac{g-1}{2}\right]^2+2\left[\frac{g-1}{2}\right]^3\right)\\ &+O\left(|D|^{7/8}(\log_q|D|)^2\right).\end{aligned}\tag{4.23}$$

Proof. The quantity S_3, that is being averaged over $D \in \mathcal{H}_{2g+1,q}$ in this proposition, is exactly equal to the quantity S_1 with the only exception that the summation index goes up to $g-1$ instead of up to g. In this way, similarly to the proof of Proposition 4.1, we can establish the result above. □

4.4. **Averaging S_4.** The main result in this section is the following proposition.

Proposition 4.10. *We have that,*

$$\begin{aligned}
&\sum_{D\in\mathcal{H}_{2g+1,q}} -4g(\log(q))^2 \sum_{m=0}^{g-1} mq^{-m/2} A(m,\chi_D) \\
&= 4g(\log(q))^2 \left(\left[\frac{g-1}{2}\right]^2 - \left[\frac{g-1}{2}\right] \right) \frac{|D|}{\zeta_A(2)} I \\
&+ \left(-1 + 2\left[\frac{g-1}{2}\right] \right) 4g(\log(q))^2 \frac{|D|}{\zeta_A(2)} (-IJ) \\
&+ 4g(\log(q))^2 \frac{|D|}{\zeta_A(2)} I(J^2-K) + O\left(|D|^{7/8} (\log_q |D|)^2 \right).
\end{aligned} \tag{4.24}$$

Proof. We prove this result in two parts. First we write

$$\begin{aligned}
&\sum_{D\in\mathcal{H}_{2g+1,q}} -4g(\log(q))^2 \sum_{m=0}^{g-1} mq^{-m/2} A(m,\chi_D) \\
&= \sum_{D\in\mathcal{H}_{2g+1,q}} -4g(\log(q))^2 \sum_{m=0}^{g-1} mq^{-m/2} \sum_{\substack{f \text{ monic} \\ \deg(f)=m \\ f=\square}} \chi_D(f) \\
&+ \sum_{D\in\mathcal{H}_{2g+1,q}} -4g(\log(q))^2 \sum_{m=0}^{g-1} mq^{-m/2} \sum_{\substack{f \text{ monic} \\ \deg(f)=m \\ f\neq\square}} \chi_D(f).
\end{aligned} \tag{4.25}$$

When f is not a perfect square we use Proposition 3.7, so we have

$$\sum_{D\in\mathcal{H}_{2g+1,q}} -4g(\log(q))^2 \sum_{m=0}^{g-1} mq^{-m/2} \sum_{\substack{f \text{ monic} \\ \deg(f)=m \\ f\neq\square}} \chi_D(f)$$

$$\ll g\sum_{m=0}^{g-1} mq^{-m/2} \sum_{\substack{f \text{ monic} \\ \deg(f)=m}} |D|^{1/2}q^{m/4} \tag{4.26}$$

$$\ll g^2|D|^{1/2}q^{3g/4}$$

$$\ll |D|^{7/8}(\log_q|D|)^2.$$

Now we treat the sum when f is a perfect square in $\mathbb{F}_q[T]$. For this we write

$$\sum_{D\in\mathcal{H}_{2g+1,q}} -4g(\log(q))^2 \sum_{m=0}^{g-1} mq^{-m/2} \sum_{\substack{f \text{ monic} \\ \deg(f)=m \\ f=\square}} \chi_D(f)$$

$$= -4g(\log(q))^2 \sum_{\substack{m=0 \\ 2|m}}^{g-1} mq^{-m/2} \sum_{\substack{f \text{ monic} \\ \deg(f)=m \\ f=\square=l^2}} \sum_{D\in\mathcal{H}_{2g+1,q}} \chi_D(l^2). \tag{4.27}$$

By using Lemma 3.2 and Lemma 3.3 we have that (4.27) becomes

$$- 4g(\log(q))^2\frac{|D|}{\zeta_A(2)} \sum_{\substack{m=0 \\ 2|m}}^{g-1} m \sum_{\substack{d \text{ monic} \\ \deg(d)\leq m/2}} \frac{\mu(d)}{|d|}\prod_{P|d}\frac{1}{|P|+1}$$

$$+ O\left(|D|^{1/2+\varepsilon}(\log_q|D|)^2\right) \tag{4.28}$$

$$= -8g(\log(q))^2\frac{|D|}{\zeta_A(2)} \sum_{\substack{d \text{ monic} \\ \deg(d)\leq[\frac{g-1}{2}]}} \frac{\mu(d)}{|d|}\prod_{P|d}\frac{1}{|P|+1} \sum_{\deg(d)\leq n\leq[\frac{g-1}{2}]} n$$

$$+ O\left(|D|^{1/2+\varepsilon}(\log_q|D|)^2\right).$$

Summing over n and grouping the powers of $\deg(d)$ we have that (4.28) is

$$
\begin{aligned}
&4g(\log(q))^2\frac{|D|}{\zeta_A(2)}\sum_{\deg(d)\le[\frac{g-1}{2}]}\frac{\mu(d)}{|d|}\prod_{P|d}\frac{1}{|P|+1}\left(\deg(d)\left(-1+2\left[\frac{g-1}{2}\right]\right)\right)\\
&+4g(\log(q))^2\frac{|D|}{\zeta_A(2)}\sum_{\deg(d)\le[\frac{g-1}{2}]}\frac{\mu(d)}{|d|}\prod_{P|d}\frac{1}{|P|+1}\left(\left[\frac{g-1}{2}\right]^2-\left[\frac{g-1}{2}\right]\right)\\
&+4g(\log(q))^2\frac{|D|}{\zeta_A(2)}\sum_{\deg(d)\le[\frac{g-1}{2}]}\frac{\mu(d)}{|d|}\prod_{P|d}\frac{1}{|P|+1}(\deg(d))^2\\
&+O\left(|D|^{1/2+\varepsilon}(\log_q|D|)^2\right).
\end{aligned}
\tag{4.29}
$$

We can rewrite the previous equation (4.29) as

$$
\begin{aligned}
&4g(\log(q))^2\frac{|D|}{\zeta_A(2)}\Big(\sum_{d\text{ monic}}-\sum_{\deg(d)>[\frac{g-1}{2}]}\Big)\frac{\mu(d)}{|d|}\prod_{P|d}\frac{1}{|P|+1}\left(\deg(d)\left(-1+2\left[\frac{g-1}{2}\right]\right)\right)\\
&+4g(\log(q))^2\frac{|D|}{\zeta_A(2)}\Big(\sum_{d\text{ monic}}-\sum_{\deg(d)>[\frac{g-1}{2}]}\Big)\frac{\mu(d)}{|d|}\prod_{P|d}\frac{1}{|P|+1}\left(\left[\frac{g-1}{2}\right]^2-\left[\frac{g-1}{2}\right]\right)\\
&+4g(\log(q))^2\frac{|D|}{\zeta_A(2)}\Big(\sum_{d\text{ monic}}-\sum_{\deg(d)>[\frac{g-1}{2}]}\Big)\frac{\mu(d)}{|d|}\prod_{P|d}\frac{1}{|P|+1}(\deg(d))^2\\
&+O\left(|D|^{1/2+\varepsilon}(\log_q|D|)^2\right).
\end{aligned}
\tag{4.30}
$$

We can bound the sums with $\deg(d)>[\frac{g-1}{2}]$ by using a simple variant of Lemma 3.4. For the sums over all monic d we use respectively Lemma 3.5 and part (i) and part (ii) of Lemma 3.6. After some arithmetic manipulations we have that (4.30) is

$$
\begin{aligned}
&\left(-1+2\left[\frac{g-1}{2}\right]\right)4g(\log(q))^2\frac{|D|}{\zeta_A(2)}(-IJ)\\
&+4g(\log(q))^2\left(\left[\frac{g-1}{2}\right]^2-\left[\frac{g-1}{2}\right]\right)\frac{|D|}{\zeta_A(2)}I\\
&+4g(\log(q))^2\frac{|D|}{\zeta_A(2)}I(J^2-K)+O(g^3|D|q^{-g/2})\\
&+O\left(|D|^{1/2+\varepsilon}(\log_q|D|)^2\right).
\end{aligned}
\tag{4.31}
$$

By combining equations (4.31) and (4.26) the proof of the proposition is complete. □

We have now all the machinery needed to prove the main result of this paper.

Proof of Theorem 2.1. The proof follows by combining the Propositions 4.1, 4.8, 4.9 and 4.10. □

Acknowledgement. This research was partially supported by EPSRC grant EP/K021132X/1.
We would like to express our gratitude to an anonymous referee whose comments improved the presentation of the paper.

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