

RUDNICK AND SOUNDARARAJAN'S THEOREM FOR FUNCTION FIELDS

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ABSTRACT. In this paper we prove a function field version of a theorem by Rudnick and Soundararajan about lower bounds for moments of quadratic Dirichlet L -functions. We establish lower bounds for the moments of quadratic Dirichlet L -functions associated to hyperelliptic curves of genus g over a fixed finite field \mathbb{F}_q in the large genus g limit.

1. INTRODUCTION

It is a fundamental problem in analytic number theory to estimate moments of central values of L -functions in families. For example, in the case of the Riemann zeta function the question is to establish asymptotic formulae for

$$(1.1) \quad M_k(T) := \int_1^T |\zeta(\tfrac{1}{2} + it)|^{2k} dt,$$

where k is a positive integer and $T \rightarrow \infty$.

A believed folklore conjecture asserts that, as $T \rightarrow \infty$, there is a positive constant C_k such that

$$(1.2) \quad M_k(T) \sim C_k T (\log T)^{k^2}.$$

Due to the work of Conrey and Ghosh [3] the conjecture above assumes a more explicit form, namely

$$(1.3) \quad C_k = \frac{a_k g_k}{\Gamma(k^2 + 1)},$$

where

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$$(1.4) \quad a_k = \prod_{p \text{ prime}} \left[\left(1 - \frac{1}{p}\right)^{k^2} \sum_{m \geq 0} \frac{d_k(m)^2}{p^m} \right],$$

g_k is an integer when k is an integer and $d_k(n)$ is the number of ways to represent n as a product of k factors.

Asymptotics for $M_k(T)$ are only known for $k = 1$, due to Hardy and Littlewood [7]

$$(1.5) \quad M_1(T) \sim T \log T,$$

and for $k = 2$, due to Ingham [10]

$$(1.6) \quad M_2(T) \sim \frac{1}{2\pi^2} T \log^4 T.$$

Unfortunately the recent technology does not allow us to obtain asymptotics for higher moments of the Riemann zeta function. The same statement applies for the higher moments of other L -functions. However, due to the precursor work of Keating and Snaith [14, 15] and, subsequently, due to the work of Conrey, Farmer, Keating, Rubinstein and Snaith [4], and Diaconu, Goldfeld and Hoffstein [5], there are now very elegant conjectures for moments of L -functions.

The work of Katz and Sarnak [12, 13] associates a symmetry group for each family of L -function and the moments are sensitive and take different forms for each one of these groups. In other words the conjectured asymptotic formulas for the moments of families of L -function depends whether the symmetry group attached to the family is unitary, orthogonal or symplectic. For a recent and detailed discussion about a working definition of a family of L -functions see [21].

We will typify the conjectures above by considering different families of L -functions. For example, the family of all Dirichlet L -functions $L(s, \chi)$, as χ varies over primitive characters (mod q), is an example of a unitary family, and it is conjectured that

$$(1.7) \quad \sum_{\chi \pmod{q}}^* |L(\frac{1}{2}, \chi)|^{2k} \sim C_{U(N)}(k) q (\log q)^{k^2},$$

where $k \in \mathbb{N}$ and $C_{U(N)}(k)$ is a positive constant. For a symplectic family of L -functions we consider the quadratic Dirichlet L -functions $L(s, \chi_d)$ associated to the quadratic character χ_d , as d varies over fundamental discriminants. In this case it is conjectured that

$$(1.8) \quad \sum_{|d| \leq X}^b L(\frac{1}{2}, \chi_d)^k \sim C_{USp(2N)}(k) X (\log X)^{k(k+1)/2},$$

where $k \in \mathbb{N}$ and $C_{USp(2N)}(k)$ is a positive constant. And finally we consider the family of L -functions associated to Hecke eigencuspforms f of weight k for the full modular group $SL(2, \mathbb{Z})$ as f varies in the set H_k of Hecke eigencuspforms. This is an example of an orthogonal family and it is conjectured that

$$(1.9) \quad \sum_{f \in H_k}^h L(\tfrac{1}{2}, f)^r \sim C_{O(N)}(r) (\log k)^{r(r-1)/2},$$

where $C_{O(N)}(r)$ is a positive constant, $k \equiv 0 \pmod{4}$ and

$$(1.10) \quad \sum_{f \in H_k}^h L(\tfrac{1}{2}, f)^r := \sum_{f \in H_k} \frac{1}{\omega_f} L(\tfrac{1}{2}, f)^r,$$

with

$$(1.11) \quad \omega_f := \frac{(4\pi)^{k-1}}{\Gamma(k-1)} \langle f, f \rangle = \frac{k-1}{2\pi^2} L(1, \text{Sym}^2 f),$$

where $\langle f, f \rangle$ denotes the Petersson inner product. For more details on Hecke eigencuspforms L -functions see Iwaniec [11].

The conjectures (1.2), (1.7) and (1.8) can be verified for small values of k and the same holds for (1.9), where it can be verified only for small values of r . Ramachandra [17] showed that

$$(1.12) \quad \int_1^T |\zeta(\tfrac{1}{2} + it)|^{2k} dt \gg T(\log T)^{k^2},$$

for positive integers k . Titchmarsh [24, Theorem 7.19] had proved a smooth version of these lower bound for positive integer k . The work of Heath-Brown [8] extends (1.12) for all positive rational numbers k . Recently Radziwiłł and Soundararajan [16] proved that

$$(1.13) \quad M_k(T) \geq e^{-30k^4} T(\log T)^{k^2},$$

for any real number $k > 1$ and all large T . For other families of L -functions, as those given above, the lower bounds for moments were proved by Rudnick and Soundararajan in [19, 20] where they have established that

$$(1.14) \quad \sum_{\substack{\chi \pmod{q} \\ \chi \neq \chi_0}}^* |L(\tfrac{1}{2}, \chi)|^{2k} \gg_k q(\log q)^{k^2},$$

for a fixed natural number k and all large primes q . They also proved in [20] that

$$(1.15) \quad \sum_{f \in H_k}^h L\left(\frac{1}{2}, f\right)^r \gg_r (\log k)^{r(r-1)/2},$$

for any given natural number r , and weight $k \geq 12$ with $k \equiv 0 \pmod{4}$. And for the symplectic family they showed that for every even natural number k

$$(1.16) \quad \sum_{|d| \leq X}^b L\left(\frac{1}{2}, \chi_d\right)^k \gg_k X (\log X)^{k(k+1)/2},$$

where the sum is taken over fundamental discriminants d . Radziwiłł and Soundararajan [16] pointed out that their method may easily be modified to provide lower bounds for moments to the case of L -functions in families, for any real number $k > 1$.

Recently, in a beautiful paper, Tamam [23] proved the function field analogue of (1.14). In this paper we consider the function field analogue of equation (1.16) for quadratic Dirichlet L -functions associated to a family of hyperelliptic curves over \mathbb{F}_q . See next section.

2. MAIN THEOREM

Before we enunciate the main theorem of this paper we need a few basic facts about rational function fields. We start by fixing a finite field \mathbb{F}_q of odd cardinality $q = p^a$ with p a prime. And we denote by $A = \mathbb{F}_q[T]$ the polynomial ring over \mathbb{F}_q and by $k = \mathbb{F}_q(T)$ the rational function field over \mathbb{F}_q .

The zeta function associated to A is defined by the following Dirichlet series

$$(2.1) \quad \zeta_A(s) := \sum_{\substack{f \in A \\ \text{monic}}} \frac{1}{|f|^s} \quad \text{for } \operatorname{Re}(s) > 1,$$

where $|f| = q^{\deg(f)}$ for $f \neq 0$ and $|f| = 0$ for $f = 0$. Surprisingly the zeta function associated to A is a much simpler object than the usual Riemann zeta function and can be showed that

$$(2.2) \quad \zeta_A(s) = \frac{1}{1 - q^{1-s}}.$$

Let D be a square-free monic polynomial in A of degree odd. Then we define the quadratic character χ_D attached to D by making use of the quadratic residue symbol for $\mathbb{F}_q[T]$ by

$$(2.3) \quad \chi_D(f) = \left(\frac{D}{f}\right).$$

In other words, if $P \in A$ is monic irreducible we have

$$(2.4) \quad \chi_D(P) = \begin{cases} 0, & \text{if } P \mid D, \\ 1, & \text{if } P \nmid D \text{ and } D \text{ is a square modulo } P, \\ -1, & \text{if } P \nmid D \text{ and } D \text{ is a non square modulo } P. \end{cases}$$

For more details about Dirichlet characters for function fields see [18, Chapter 3] and [6].

We attach to the character χ_D the quadratic Dirichlet L -function defined by

$$(2.5) \quad L(s, \chi_D) = \sum_{\substack{f \in A \\ f \text{ monic}}} \frac{\chi_D(f)}{|f|^s} \quad \text{for } \operatorname{Re}(s) > 1.$$

If $D \in \mathcal{H}_{2g+1,q}$, where

$$(2.6) \quad \mathcal{H}_{2g+1,q} = \{D \in A, \text{ square-free, monic and } \deg(D) = 2g + 1\},$$

then the L -function associated to χ_D is indeed the numerator of the zeta function associated to the hyperelliptic curve $C_D : y^2 = D(T)$ and therefore $L(s, \chi_D)$ is a polynomial in $u = q^{-s}$ of degree $2g$ given by

$$(2.7) \quad L(s, \chi_D) = \sum_{n=0}^{2g} \sum_{\substack{f \text{ monic} \\ \deg(f)=n}} \chi_D(f) q^{-ns}.$$

(see [18, Propositions 14.6 and 17.7] and [1, Section 3]).

This L -function satisfies a functional equation, namely

$$(2.8) \quad L(s, \chi_D) = (q^{1-2s})^g L(1-s, \chi_D),$$

and the Riemann hypothesis for curves proved by Weil [25] tell us that all the zeros of $L(s, \chi_D)$ have real part equals $1/2$.

The main result of this paper is now presented:

Theorem 2.1. *For every even natural number k we have,*

$$(2.9) \quad \frac{1}{\#\mathcal{H}_{2g+1,q}} \sum_{D \in \mathcal{H}_{2g+1,q}} L\left(\frac{1}{2}, \chi_D\right)^k \gg_k (\log_q |D|)^{k(k+1)/2}.$$

Remark 2.1. *To avoid any misunderstanding concerning the notation and conventions presented in this paper it is necessary a note about the notation used in the theorem above and in the rest of this note. On the formula above the right-hand side of the main lower bound appears $|D| = q^{2g+1}$ while D is the summation variable on the left-hand side of that same formula. This is done because the function $D \mapsto |D|$ is constant within $\mathcal{H}_{2g+1,q}$ and so we can always write*

$$\sum_{D \in \mathcal{H}_{2g+1,q}} |D| = |\mathcal{H}_{2g+1,q}| \sum_{D \in \mathcal{H}_{2g+1,q}} 1.$$

In the case the reader feel uncomfortable with the above notation he/she can always remember that $|\mathcal{H}_{2g+1,q}| = q^{2g+1}$.

Remark 2.2. For simplicity, we will restrict ourselves to the fundamental discriminants $D \in A$, D monic and $\deg(D) = 2g + 1$. But the calculations are analogous for the even case, i.e., $\deg(D) = 2g + 2$.

Using the same techniques developed by Rudnick and Soundararajan in [19, 20] and extended for function fields in this paper we can also prove the following theorem.

Theorem 2.2. For every even natural number k and $n = 2g+1$ or $n = 2g+2$ we have,

$$(2.10) \quad \frac{1}{\pi_A(n)} \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=n}} L\left(\frac{1}{2}, \chi_P\right)^k \gg_k (\log_q |P|)^{\frac{k(k+1)}{2}},$$

where $\pi_A(n) = \#\{P \in \mathbb{F}_q[T] \text{ monic and irreducible, } \deg(P) = n\}$ and the prime number theorem for polynomials [18, Theorem 2.2] says that $\pi_A(n) = \frac{q^n}{n} + O\left(\frac{q^{n/2}}{n}\right)$.

3. NECESSARY TOOLS

In this section we present some auxiliary lemmas that will be used in the proof of the main theorem. We start with:

Lemma 3.1 (“Approximate” Functional Equation). *Let $D \in \mathcal{H}_{2g+1,q}$. Then $L(s, \chi_D)$ can be represented as*

$$(3.1) \quad L(s, \chi_D) = \sum_{\substack{f_1 \text{ monic} \\ \deg(f_1) \leq g}} \frac{\chi_D(f_1)}{|f_1|^s} + (q^{1-2s})^g \sum_{\substack{f_2 \text{ monic} \\ \deg(f_2) \leq g-1}} \frac{\chi_D(f_2)}{|f_2|^{1-s}}.$$

Proof. The proof of this Lemma can be found in [1, Lemma 3.3]. □

The following lemma is the function field analogue of Pólya–Vinogradov inequality for character sums.

Lemma 3.2 (Pólya–Vinogradov inequality for $\mathbb{F}_q(T)$). *Let χ be a non-principal Dirichlet character modulo $Q \in \mathbb{F}_q[T]$ such that $\deg(Q)$ is odd. Then we have,*

$$(3.2) \quad \sum_{\deg(f)=x} \chi(f) \ll |Q|^{1/2}.$$

Proof. The proof of this Lemma can be found in [9, Proposition 2.1]. \square

The next lemma is taken from Andrade-Keating [1, Proposition 5.2] and it is about counting the number of square-free polynomials coprime to a fixed monic polynomial.

Lemma 3.3. *Let $f \in A$ be a fixed monic polynomial. Then for all $\varepsilon > 0$ we have that*

$$(3.3) \quad \sum_{\substack{D \in \mathcal{H}_{2g+1,q} \\ (D,f)=1}} 1 = \frac{|D|}{\zeta_A(2)} \prod_{\substack{P \text{ monic} \\ \text{irreducible} \\ P|f}} \left(\frac{|P|}{|P|+1} \right) + O\left(|D|^{\frac{1}{2}}|f|^\varepsilon\right).$$

4. PROOF OF THEOREM 2.1

In this section we prove Theorem 2.1.

Let k be a given even number, and set $x = \frac{2(2g)}{15k}$. We define

$$(4.1) \quad A(D) = \sum_{\deg(n) \leq x} \frac{\chi_D(n)}{\sqrt{|n|}},$$

and let

$$(4.2) \quad S_1 := \sum_{D \in \mathcal{H}_{2g+1,q}} L\left(\frac{1}{2}, \chi_D\right) A(D)^{k-1},$$

and

$$(4.3) \quad S_2 := \sum_{D \in \mathcal{H}_{2g+1,q}} A(D)^k.$$

An application of Triangle inequality followed by Hölder's inequality gives us that,

$$(4.4) \quad \begin{aligned} & \left| \sum_{D \in \mathcal{H}_{2g+1,q}} L\left(\frac{1}{2}, \chi_D\right) A(D)^{k-1} \right| \leq \sum_{D \in \mathcal{H}_{2g+1,q}} |L\left(\frac{1}{2}, \chi_D\right)| |A(D)|^{k-1} \\ & \leq \left(\sum_{D \in \mathcal{H}_{2g+1,q}} L\left(\frac{1}{2}, \chi_D\right)^k \right)^{1/k} \left(\sum_{D \in \mathcal{H}_{2g+1,q}} A(D)^k \right)^{\frac{k-1}{k}}. \end{aligned}$$

From (4.4) we have

$$\begin{aligned}
\sum_{D \in \mathcal{H}_{2g+1,q}} L\left(\frac{1}{2}, \chi_D\right)^k &\geq \frac{\left(\sum_{D \in \mathcal{H}_{2g+1,q}} L\left(\frac{1}{2}, \chi_D\right) A(D)^{k-1}\right)^k}{\left(\sum_{D \in \mathcal{H}_{2g+1,q}} A(D)^k\right)^{k-1}} \\
(4.5) \qquad \qquad \qquad &= \frac{S_1^k}{S_2^{k-1}}.
\end{aligned}$$

Hence from (4.5) we can see that to prove Theorem 2.1 we only need to give satisfactory estimates for S_1 and S_2 . We start with S_2 .

4.1. **Estimating S_2 .** We have that

$$(4.6) \qquad A(D)^k = \sum_{\substack{n_1, \dots, n_k \\ \deg(n_j) \leq x \\ j=1, \dots, k}} \frac{\chi_D(n_1 \dots n_k)}{\sqrt{|n_1| \dots |n_k|}}.$$

So,

$$\begin{aligned}
S_2 &= \sum_{D \in \mathcal{H}_{2g+1,q}} A(D)^k \\
(4.7) \qquad &= \sum_{\substack{n_1, \dots, n_k \\ \deg(n_j) \leq x \\ j=1, \dots, k}} \frac{1}{\sqrt{|n_1| \dots |n_k|}} \sum_{D \in \mathcal{H}_{2g+1,q}} \left(\frac{D}{n_1 \dots n_k}\right).
\end{aligned}$$

At this stage we need an auxiliary Lemma. It is called orthogonal relations for quadratic characters and it has appeared in a different form in [1, 2, 6].

Lemma 4.1. *If $n \in A$ is not a perfect square then*

$$(4.8) \qquad \sum_{\substack{D \in \mathcal{H}_{2g+1,q} \\ n \neq \square}} \left(\frac{D}{n}\right) \ll |D|^{1/2} |n|^{1/4}.$$

And if $n \in A$ is a perfect square then

$$(4.9) \qquad \sum_{\substack{D \in \mathcal{H}_{2g+1,q} \\ n = \square}} \left(\frac{D}{n}\right) = \frac{|D|}{\zeta_A(2)} \prod_{\substack{P \text{ monic} \\ \text{irreducible} \\ P|n}} \left(\frac{|P|}{|P|+1}\right) + O\left(|D|^{1/2} |n|^\varepsilon\right),$$

for any $\varepsilon > 0$.

Remark 4.2. *Equation (4.8) can be seen as an improvement on the estimate given in [6, Lemma 3.1]. And the same equation (4.8) can be used to improve the error term in the first moment of quadratic Dirichlet L -functions over function fields as given in [1, Theorem 2.1].*

Proof. If $n = \square$, then

$$(4.10) \quad \sum_{\substack{D \in \mathcal{H}_{2g+1,q} \\ n = \square}} \left(\frac{D}{n} \right) = \sum_{\substack{D \in \mathcal{H}_{2g+1,q} \\ (D,n)=1}} 1.$$

By invoking Lemma 3.3 we establish equation (4.9).

For (4.8) we write

$$(4.11) \quad \begin{aligned} \sum_{D \in \mathcal{H}_{2g+1,q}} \left(\frac{D}{n} \right) &= \sum_{2\alpha + \beta = 2g+1} \sum_{\deg(B) = \beta} \sum_{\deg(A) = \alpha} \mu(A) \left(\frac{A^2 B}{n} \right) \\ &= \sum_{0 \leq \alpha \leq g} \sum_{\deg(A) = \alpha} \mu(A) \left(\frac{A^2}{n} \right) \sum_{\deg(B) = 2g+1-2\alpha} \left(\frac{B}{n} \right) \\ &\leq \sum_{0 \leq \alpha \leq g} \sum_{\deg(A) = \alpha} \sum_{\deg(B) = 2g+1-2\alpha} \left(\frac{B}{n} \right). \end{aligned}$$

If $n \neq \square$ then $\sum_{\deg(B) = 2g+1-2\alpha} \left(\frac{B}{n} \right)$ is a character sum to a non-principal character modulo n . So using Lemma 3.2 we have that

$$(4.12) \quad \sum_{\deg(B) = 2g+1-2\alpha} \left(\frac{B}{n} \right) \ll |n|^{1/2}.$$

Further we can estimate trivially the non-principal character sum by

$$(4.13) \quad \sum_{\deg(B) = 2g+1-2\alpha} \left(\frac{B}{n} \right) \ll \frac{|D|}{|A|^2} = q^{2g+1-2\alpha}.$$

Thus, if $n \neq \square$, we obtain that

$$(4.14) \quad \begin{aligned} \sum_{D \in \mathcal{H}_{2g+1,q}} \left(\frac{D}{n} \right) &\ll \sum_{0 \leq \alpha \leq g} \sum_{\deg(A) = \alpha} \min \left(|n|^{1/2}, \frac{|D|}{|A|^2} \right) \\ &\ll |D|^{\frac{1}{2}} |n|^{\frac{1}{4}}, \end{aligned}$$

upon using the first bound (4.12) for $\alpha \leq g - \frac{\deg(n)}{4}$ and the second bound (4.13) for larger α . And this concludes the proof of the lemma. \square

Using Lemma 4.1 in (4.7) we obtain that

$$\begin{aligned}
(4.15) \quad S_2 &= \sum_{\substack{n_1, \dots, n_k \\ \deg(n_j) \leq x \\ j=1, \dots, k \\ n_1 \dots n_k = \square}} \frac{1}{\sqrt{|n_1| \dots |n_k|}} \left(\frac{|D|}{\zeta_A(2)} \prod_{\substack{P \text{ monic} \\ \text{irreducible} \\ P|n_1 \dots n_k}} \left(\frac{|P|}{|P|+1} \right) \right) \\
&+ \sum_{\substack{n_1, \dots, n_k \\ \deg(n_j) \leq x \\ j=1, \dots, k \\ n_1 \dots n_k = \square}} \frac{1}{\sqrt{|n_1| \dots |n_k|}} O \left(|D|^{\frac{1}{2}} |n_1 \dots n_k|^\varepsilon \right) \\
&+ \sum_{\substack{n_1, \dots, n_k \\ \deg(n_j) \leq x \\ j=1, \dots, k \\ n_1 \dots n_k \neq \square}} \frac{1}{\sqrt{|n_1| \dots |n_k|}} O \left(|D|^{\frac{1}{2}} |n_1 \dots n_k|^{\frac{1}{4}} \right).
\end{aligned}$$

After some arithmetic manipulations with the O -terms we get that

$$\begin{aligned}
(4.16) \quad S_2 &= \frac{|D|}{\zeta_A(2)} \sum_{\substack{n_1, \dots, n_k \\ \deg(n_j) \leq x \\ j=1, \dots, k \\ n_1 \dots n_k = \square}} \frac{1}{\sqrt{|n_1| \dots |n_k|}} \prod_{\substack{P \text{ monic} \\ \text{irreducible} \\ P|n_1 \dots n_k}} \left(\frac{|P|}{|P|+1} \right) \\
&+ O \left(|D|^{\frac{1}{2}} q^{\left(\frac{3}{4} + \varepsilon\right)x} \right).
\end{aligned}$$

Since $x = \frac{2(2g)}{15k}$, the error term above is $\ll |D|^{\frac{3}{5}}$. So,

$$(4.17) \quad S_2 = \frac{|D|}{\zeta_A(2)} \sum_{\substack{n_1, \dots, n_k \\ \deg(n_j) \leq x \\ j=1, \dots, k \\ n_1 \dots n_k = \square}} \frac{1}{\sqrt{|n_1| \dots |n_k|}} \prod_{\substack{P \text{ monic} \\ \text{irreducible} \\ P|n_1 \dots n_k}} \left(\frac{|P|}{|P|+1} \right) + O \left(|D|^{\frac{3}{5}} \right).$$

Writing $n_1 \dots n_k = m^2$ we see that

$$\begin{aligned}
 & \sum_{\substack{m^2 \text{ monic} \\ \deg(m^2) \leq x}} \frac{d_k(m^2)}{|m|} \prod_{\substack{P \text{ monic} \\ \text{irreducible} \\ P|m}} \left(\frac{|P|}{|P|+1} \right) \\
 & \leq \sum_{\substack{n_1, \dots, n_k \\ \deg(n_j) \leq x \\ j=1, \dots, k \\ n_1 \dots n_k = \square = m^2}} \frac{1}{\sqrt{|n_1| \dots |n_k|}} \prod_{\substack{P \text{ monic} \\ \text{irreducible} \\ P|n_1 \dots n_k}} \left(\frac{|P|}{|P|+1} \right) \\
 (4.18) \quad & \leq \sum_{\substack{m^2 \text{ monic} \\ \deg(m^2) \leq kx}} \frac{d_k(m^2)}{|m|} \prod_{\substack{P \text{ monic} \\ \text{irreducible} \\ P|m}} \left(\frac{|P|}{|P|+1} \right),
 \end{aligned}$$

where $d_k(m)$ represents the number of ways to write the monic polynomial m as a product of k factors.

We need to obtain an estimate for

$$(4.19) \quad \sum_{\substack{m \text{ monic} \\ \deg(m)=x}} d_k(m^2) a_m,$$

where $a_m = \prod_{\substack{P \text{ monic} \\ \text{irreducible} \\ P|m}} \left(\frac{|P|}{|P|+1} \right)$.

To obtain the desired estimate we consider the corresponding Dirichlet series

$$(4.20) \quad \zeta_f(s) = \sum_{m \text{ monic}} \frac{d_k(m^2) a_m}{|m|^s} = \sum_{n=0}^{\infty} \sum_{\deg(m)=x} d_k(m^2) a_m u^x = Z_f(u),$$

with $u = q^{-s}$. Writing the above as an Euler product

$$(4.21) \quad \zeta_f(s) = \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left(1 + \frac{d_k(P^2) a_P}{|P|^s} + \frac{d_k(P^4) a_{P^2}}{|P|^{2s}} + \dots \right),$$

we can identify the poles of $\zeta_f(s)$. Similar calculations carried out in the classical case by Soundararajan and Rudnick [20, page 9] and Selberg [22, Theorem 2], and for function fields by Andrade and Keating [2, Section 4.3] shows us that $\zeta_f(s)$ has a pole at $s = 1$ of order $\frac{k(k+1)}{2}$. Therefore we can write

$$\begin{aligned}
(4.22) \quad \zeta_f(s) &= \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left(1 - \frac{1}{|P|^s}\right)^{-\frac{k(k+1)}{2}} \\
&\times \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left(1 + \left(\frac{|P|}{|P|+1} \sum_{j=1}^{\infty} \frac{d_k(P^{2j})}{|P|^{js}}\right)\right) \left(1 - \frac{1}{|P|^s}\right)^{\frac{k(k+1)}{2}},
\end{aligned}$$

where the first product has a pole at $s = 1$ of order $\frac{k(k+1)}{2}$ and the second product above (4.22) is convergent for $\operatorname{Re}(s) > 1$ and holomorphic in $\{s \in B \mid \operatorname{Re}(s) = 1\}$ with

$$(4.23) \quad B = \left\{s \in \mathbb{C} \mid -\frac{\pi i}{\log(q)} \leq \Im(s) < \frac{\pi i}{\log(q)}\right\}.$$

Thus we can use Theorem 17.4 from [18] to obtain the desired estimate. But we sketch below how this can be done. A standard contour integration (Cauchy's theorem)

$$(4.24) \quad \frac{1}{2\pi i} \oint_{C_\varepsilon + C} \frac{Z_f(u)}{u^{x+1}} du = \sum \operatorname{Res}(Z_f(u)u^{-x-1}),$$

where C is the boundary of the disc $\{u \in \mathbb{C} \mid |u| \leq q^{-\delta}\}$ for some $\delta < 1$ and C_ε a small circle about $s = 0$ oriented clockwise. There is only one pole in the integration region $C_\varepsilon + C$ and it is located at $u = q^{-1}$ as can be seen from (4.22). To find the residue there, we expand both $Z_f(u)$ and u^{-x-1} in Laurent series about $u = q^{-1}$, multiply the results together, and pick out the coefficient of $(u - q^{-1})^{-1}$. After this residue calculation we obtain that

$$(4.25) \quad \sum_{\substack{m \text{ monic} \\ \deg(m)=x}} d_k(m^2)a_m \sim C(k)q^x x^{\frac{k(k+1)}{2}-1},$$

for a positive constant $C(k)$ explicitly given by

$$(4.26) \quad C(k) = \frac{\log(q)^{\frac{k(k+1)}{2}}}{\left(\frac{k(k+1)}{2} - 1\right)!} \alpha,$$

with

$$(4.27) \quad \alpha = \lim_{s \rightarrow 1} \left[(s-1)^{\frac{k(k+1)}{2}} \zeta_f(s) \right].$$

In the end we obtain that

$$(4.28) \quad \sum_{\substack{m \text{ monic} \\ \deg(m) \leq z}} \frac{d_k(m^2)}{|m|} \prod_{\substack{P \text{ monic} \\ P|m}} \left(\frac{|P|}{|P|+1} \right) \sim C(k)(z)^{k(k+1)/2}.$$

Therefore we can conclude that

$$(4.29) \quad S_2 \ll |D|(\log_q |D|)^{k(k+1)/2}.$$

4.2. Estimating S_1 . It remains to evaluate S_1 and for that we need an “approximate” functional equation for $L(\frac{1}{2}, \chi_D)$. Using Lemma 3.1 with $s = \frac{1}{2}$ we have that

$$(4.30) \quad \begin{aligned} S_1 &= \sum_{D \in \mathcal{H}_{2g+1,q}} \left(\sum_{\deg(f_1) \leq g} \frac{\chi_D(f_1)}{|f_1|^{1/2}} + \sum_{\deg(f_2) \leq g-1} \frac{\chi_D(f_2)}{|f_2|^{1/2}} \right) \\ &\quad \times \left(\sum_{\substack{n_1, \dots, n_{k-1} \\ \deg(n_j) \leq x \\ j=1, \dots, k-1}} \frac{\chi_D(n_1 \dots n_{k-1})}{\sqrt{|n_1| \dots |n_{k-1}|}} \right) \\ &= \sum_{\deg(f_1) \leq g} \frac{1}{\sqrt{|f_1|}} \sum_{\substack{n_1, \dots, n_{k-1} \\ \deg(n_j) \leq x \\ j=1, \dots, k-1}} \frac{1}{\sqrt{|n_1| \dots |n_{k-1}|}} \sum_{D \in \mathcal{H}_{2g+1,q}} \left(\frac{D}{f_1 n_1 \dots n_{k-1}} \right) \\ &\quad + \sum_{\deg(f_2) \leq g-1} \frac{1}{\sqrt{|f_2|}} \sum_{\substack{n_1, \dots, n_{k-1} \\ \deg(n_j) \leq x \\ j=1, \dots, k-1}} \frac{1}{\sqrt{|n_1| \dots |n_{k-1}|}} \sum_{D \in \mathcal{H}_{2g+1,q}} \left(\frac{D}{f_2 n_1 \dots n_{k-1}} \right). \end{aligned}$$

In the last equality in equation (4.30) the sums over f_1 and f_2 are exactly the same, with the only difference being the size of the sums, i.e., $\deg(f_1) \leq g$ and $\deg(f_2) \leq g-1$. We estimate only the f_1 sum in the last equality and the result being the same for the f_2 sum just replacing g by $g-1$.

If $f_1 n_1 \dots n_{k-1}$ is not a square then an application of Lemma 4.1 gives us that

$$\begin{aligned}
& \sum_{\deg(f_1) \leq g} \frac{1}{\sqrt{|f_1|}} \sum_{\substack{n_1, \dots, n_{k-1} \\ \deg(n_j) \leq x \\ j=1, \dots, k-1}} \frac{1}{\sqrt{|n_1| \dots |n_{k-1}|}} \sum_{D \in \mathcal{H}_{2g+1, q}} \left(\frac{D}{f_1 n_1 \dots n_{k-1}} \right) \\
& \ll \sum_{\deg(f_1) \leq g} \frac{1}{\sqrt{|f_1|}} \sum_{\substack{n_1, \dots, n_{k-1} \\ \deg(n_j) \leq x \\ j=1, \dots, k-1}} \frac{1}{\sqrt{|n_1| \dots |n_{k-1}|}} |D|^{\frac{1}{2}} |f_1 n_1 \dots n_{k-1}|^{\frac{1}{4}} \\
& = |D|^{\frac{1}{2}} \sum_{\deg(f_1) \leq g} |f_1|^{-\frac{1}{4}} \sum_{\deg(n_1) \leq x} |n_1|^{-\frac{1}{4}} \dots \sum_{\deg(n_{k-1}) \leq x} |n_{k-1}|^{-\frac{1}{4}} \\
(4.31) \quad & \ll |D|^{\frac{1}{2}} q^{\frac{3}{4}g} (q^x)^{\frac{3}{4}(k-1)}.
\end{aligned}$$

With our choice of x , we have that for $f_1 n_1 \dots n_{k-1}$ not a square

$$\begin{aligned}
& \sum_{\deg(f_1) \leq g} \frac{1}{\sqrt{|f_1|}} \sum_{\substack{n_1, \dots, n_{k-1} \\ \deg(n_j) \leq x \\ j=1, \dots, k-1}} \frac{1}{\sqrt{|n_1| \dots |n_{k-1}|}} \sum_{D \in \mathcal{H}_{2g+1, q}} \left(\frac{D}{f_1 n_1 \dots n_{k-1}} \right) \\
(4.32) \quad & \ll |D|^{\frac{39}{40}}.
\end{aligned}$$

For $f_2 n_1 \dots n_{k-1}$ not equal to a perfect square, the same reasoning gives

$$\begin{aligned}
& \sum_{\deg(f_2) \leq g-1} \frac{1}{\sqrt{|f_2|}} \sum_{\substack{n_1, \dots, n_{k-1} \\ \deg(n_j) \leq x \\ j=1, \dots, k-1}} \frac{1}{\sqrt{|n_1| \dots |n_{k-1}|}} \sum_{D \in \mathcal{H}_{2g+1, q}} \left(\frac{D}{f_2 n_1 \dots n_{k-1}} \right) \\
(4.33) \quad & \ll |D|^{\frac{39}{40}}.
\end{aligned}$$

It remains to estimate the main-term in S_1 . If $f_1 n_1 \dots n_{k-1}$ is a perfect square then

$$\begin{aligned}
(4.34) \quad & \sum_{D \in \mathcal{H}_{2g+1, q}} \sum_{\substack{f_1, n_1, \dots, n_{k-1} \\ \deg(f_1) \leq g \\ \deg(n_j) \leq x \\ j=1, \dots, k-1}} \frac{1}{\sqrt{|f_1| |n_1| \dots |n_{k-1}|}} \chi_D(f_1 n_1 \dots n_{k-1}) \\
& = \sum_{\substack{f_1, n_1, \dots, n_{k-1} \\ \deg(f_1) \leq g \\ \deg(n_j) \leq x \\ j=1, \dots, k-1}} \frac{1}{\sqrt{|f_1| |n_1| \dots |n_{k-1}|}} \sum_{D \in \mathcal{H}_{2g+1, q}} \chi_D(f_1 n_1 \dots n_{k-1}).
\end{aligned}$$

By Lemma 3.3 we have that (4.34) becomes

$$\begin{aligned}
 & \sum_{D \in \mathcal{H}_{2g+1,q}} \sum_{\substack{f_1, n_1, \dots, n_{k-1} \\ \deg(f_1) \leq g \\ \deg(n_j) \leq x \\ j=1, \dots, k-1}} \frac{1}{\sqrt{|f_1| |n_1| \dots |n_{k-1}|}} \chi_D(f_1 n_1 \dots n_{k-1}) \\
 &= \sum_{\substack{f_1, n_1, \dots, n_{k-1} \\ \deg(f_1) \leq g \\ \deg(n_j) \leq x \\ j=1, \dots, k-1}} \frac{1}{\sqrt{|f_1| |n_1| \dots |n_{k-1}|}} \frac{|D|}{\zeta_A(2)} \prod_{\substack{P \text{ monic} \\ \text{irreducible} \\ P|f_1 n_1 \dots n_{k-1}}} \left(\frac{|P|}{|P|+1} \right) \\
 (4.35) \quad & + O \left(\sum_{\substack{f_1, n_1, \dots, n_{k-1} \\ \deg(f_1) \leq g \\ \deg(n_j) \leq x \\ j=1, \dots, k-1}} \frac{1}{\sqrt{|f_1| |n_1| \dots |n_{k-1}|}} |D|^{\frac{1}{2}} |f_1 n_1 \dots n_{k-1}|^\varepsilon \right)
 \end{aligned}$$

If we call $a_f = \prod_{P|f} \left(\frac{|P|}{|P|+1} \right)$, then we have

$$\begin{aligned}
 & \sum_{D \in \mathcal{H}_{2g+1,q}} \sum_{\substack{f_1, n_1, \dots, n_{k-1} \\ \deg(f_1) \leq g \\ \deg(n_j) \leq x \\ j=1, \dots, k-1}} \frac{1}{\sqrt{|f_1| |n_1| \dots |n_{k-1}|}} \chi_D(f_1 n_1 \dots n_{k-1}) \\
 (4.36) \quad &= \frac{|D|}{\zeta_A(2)} \sum_{\deg(m) \leq \frac{g+(k-1)x}{2}} \frac{a_{m^2} d_k(m^2)}{|m|} + O \left(|D|^{\frac{1}{2}} q^{g(\varepsilon - \frac{1}{2}) + g} (q^{x(\varepsilon - \frac{1}{2}) + x})^{k-1} \right).
 \end{aligned}$$

With our choice of x we have that the O -term above is $\ll |D|^{39/40}$.

The last step is to estimate the main term contribution

$$(4.37) \quad \frac{|D|}{\zeta_A(2)} \sum_{\deg(m) \leq \frac{g+(k-1)x}{2}} \frac{a_{m^2} d_k(m^2)}{|m|}.$$

By employing the same reasoning of Rudnick and Soundararajan [20, page 10] we write $n_1 \dots n_{k-1} = rh^2$ where r and h are monic polynomials and r is square-free. Then f_1 is of the form rl^2 . With this notation the main term contribution is

$$(4.38) \quad \frac{|D|}{\zeta_A(2)} \sum_{\substack{n_1, \dots, n_{k-1} \\ n_1 \dots n_{k-1} = rh^2 \\ \deg(n_j) \leq x \\ j=1, \dots, k-1}} \frac{1}{|rh|} \sum_{\substack{l \text{ monic} \\ \deg(l) \leq \frac{g - \deg(r)}{2}}} \frac{1}{|l|} a_{rhl}.$$

Note that $\deg(r) \leq (k-1)x$ and an easy calculation as those used in [20, page 10] and [1, Lemma 5.7 and pages 2812–2813] gives that the sum over l above is

$$(4.39) \quad \sum_{\substack{l \text{ monic} \\ \deg(l) \leq \frac{g-\deg(r)}{2}}} \frac{1}{|l|} a_{rhl} \sim C(r, h) a_{rh} g,$$

for some positive constant $C(r, h)$.

Therefore follows that the main term contribution to (4.36) is

$$(4.40) \quad \begin{aligned} &\gg |D|(\log_q |D|) \sum_{\substack{n_1, \dots, n_{k-1} \\ n_1 \cdots n_{k-1} = rh^2 \\ \deg(n_j) \leq x \\ j=1, \dots, k-1}} \frac{1}{|rh|} a_{rh} \\ &\gg |D|(\log_q |D|) \sum_{\substack{r, h \\ \deg(rh^2) \leq x}} \frac{d_{k-1}(rh^2)}{|rh|} a_{rh} \\ &\gg |D|(\log_q |D|)^{k(k+1)/2}, \end{aligned}$$

where the last bound follows by activating the same estimate as proved in past section, replacing k by $k-1$. The same argument applies to the second sum in (4.29) replacing g by $g-1$. Therefore we can conclude that

$$(4.41) \quad S_1 \gg |D|(\log_q |D|)^{k(k+1)/2}.$$

Combining (4.29) and (4.41) finishes the prove of Theorem 2.1.

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