The First Moment of $L(\frac{1}{2},\chi)$ for Real Quadratic Function Fields

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ABSTRACT: In this paper we use techniques first introduced by Florea to improve the asymptotic formula for the first moment of the quadratic Dirichlet L-functions over the rational function field, running over all monic, square-free polynomials of even degree at the central point. With some extra technical difficulties that do not appear in Florea's paper, we prove that there is an extra main term of size $gq^{\frac{2g+2}{3}}$, whilst bounding the error term by $q^{\frac{g}{2}(1+\epsilon)}$.

1 Introduction

An important and well-studied problem in analytic number theory is to understand the asymptotic behaviour of moments of families of L-functions. Considering the family of Dirichlet L-functions, $L(s,\chi_d)$, with χ_d a real primitive Dirichlet character modulo d defined by the Jacobi symbol $\chi_d(n) = \left(\frac{d}{n}\right)$, a problem is to understand the asymptotic behaviour of

$$\sum_{0<d\leq D} L\left(s,\chi_d\right)^k, \tag{1.1}$$

summing over fundamental discriminants d, as $D\to\infty$. In this context, Jutila [16], proved, when $s=\frac{1}{2}$, that

$$\sum_{0<d\leq D} L\left(\frac{1}{2},\chi_d\right) = \frac{P(1)}{4\zeta(2)}\left[\log\left(\frac{D}{\pi}\right)+\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}\right)+4\gamma-1+4\frac{P'}{P}(1)\right]+O(D^{\frac{3}{4}+\epsilon}), \tag{1.2}$$

for all $\epsilon>0$ and

$$P(s)=\prod_P\left(1-\frac{1}{p^s(p+1)}\right).$$

Goldfeld and Hoffstein [12], improved the error term to $D^{\frac{19}{32}+\epsilon}$. Young [25], showed that the error term is bounded by $D^{\frac{1}{2}+\epsilon}$ when considering the smoothed first moment. Jutila [16], computed the second moment and Soundararajan [23], computed the second and third moments, when averaging over real, primitive, even characters with conductors $8d$. It is conjectured that

$$\sum_{0<d\leq D} L\left(\frac{1}{2},\chi_d\right)^k \sim C_k D(\log D)^{\frac{k(k+1)}{2}},$$

where the sum is over fundamental discriminants. Keating and Snaith [19], conjectured a precise value for C_k and Conrey *et al* [8], conjectured the integral moments and formulas for the principal

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lower order terms.

In the function field setting, the analogue problem is to understand the asymptotic behaviour of

$$\sum_{D\in\mathcal{H}_{2g+1}} L(s,\chi_D)^k \tag{1.3}$$

as $|D| = q^{\deg(D)} \to \infty$, where $\mathcal{H}_{2g+1}$ denotes the space of monic, square-free polynomials of degree $2g+1$ over $\mathbb{F}_q[x]$, which corresponds to the imaginary quadratic function field, and $L(s,\chi_D)$ denotes the quadratic Dirichlet L-function for the rational function field. Since we are letting $|D| \to \infty$, there are two limits to consider. The first is to fix g and let $q \to \infty$ and the second is to fix q and let $g \to \infty$. Katz and Sarnak [17, 18] used equidistribution results to relate the q limit of (1.3) to a random matrix theory integral, which was then computed by Keating and Snaith [19]. Therefore we will concentrate on the other limit, namely when q is fixed and we let $g \to \infty$. In this context, Andrade and Keating [3], computed the first moment of (1.3), when $s = \frac{1}{2}$, which is considered to be the function field analogue of Jutila's result (1.2). In particular they proved the following result.

Theorem 1.1 (Andrade and Keating)**.** *Let q be the fixed cardinality of the ground field and assume for simplicity that $q \equiv 1(mod\ 4)$. Then*

$$\sum_{D\in\mathcal{H}_{2g+1}} L\left(\frac{1}{2},\chi_D\right) = \frac{P(1)}{2\zeta_{\mathbb{A}}(2)}|D|\left[\log_q|D| + 1 + \frac{4}{\log q}\frac{P'}{P}(1)\right] + O(|D|^{\frac{3}{4}+\frac{\log_q 2}{2}}), \tag{1.4}$$

where

$$P(s) = \prod_P\left(1 - \frac{1}{|P|^s(|P|+1)}\right)$$

and $\zeta_{\mathbb{A}}$ denotes the zeta function associated with $\mathbb{A} = \mathbb{F}_q[x]$.

Andrade and Keating [4], also conjectured asymptotic formulas for higher and integral moments for (1.3) which is considered to be the function field analogue of Keating and Snaith's result [19] and Conrey *et al* [8]. Rubinstein and Wu [21], provided numerical evidence for the conjecture given by Andrade and Keating [4]. They numerically computed the moments for $k \le 10$ and $d \le 18$, where $d = 2g+1$, for various values of q and compared them to the conjectured formulas.

Using a similar method to Young's [25], in the number field case, Florea [9], improved the asymptotic formula obtained by Andrade and Keating (1.4) by obtaining a secondary main term of size $gq^{\frac{2g+1}{3}}$ and bounding the error term by $q^{\frac{g}{2}(1+\epsilon)}$.

Theorem 1.2 (Florea)**.** *Let q be a prime with $q \equiv 1(mod\ 4)$. Then*

$$\sum_{D\in\mathcal{H}_{2g+1}} L\left(\frac{1}{2},\chi_D\right) = \frac{P(1)}{2\zeta_{\mathbb{A}}(2)}q^{2g+1}\left[(2g+1) + 1 + \frac{4}{\log q}\frac{P'}{P}(1)\right] + q^{\frac{2g+1}{3}}R(2g+1) + O(q^{\frac{g}{2}(1+\epsilon)}), \tag{1.5}$$

where R is a polynomial of degree 1 that can be explicitly calculated.

Using a similar method, Florea [10, 11], computed the second, third and fourth moments of (1.3) at $s = \frac{1}{2}$ and showed that these asymptotic formulas agree with the formulas conjectured by Andrade and Keating in [4].

In [1], Andrade obtained an asymptotic formula for the first moment of (1.3) at $s = 1$. In particular, he proved that

$$\sum_{D\in\mathcal{H}_{2g+1}} L(1,\chi_D) = |D|P(2) + O((2q)^g). \tag{1.6}$$

Using the techniques presented by Florea, Andrade and Jung [2], improved the asymptotic formula, (1.6), by obtaining a secondary main term of size $q^{\frac{g}{3}}$ and bounding the error term by $q^{g\epsilon}$, for any $\epsilon > 0$. In particular, they proved that

$$\sum_{D\in\mathcal{H}_{2g+1}} L(1,\chi_D) = P(2)q^{2g+1} + c_1 q^{\frac{g}{3}} + O(q^{g\epsilon}), \tag{1.7}$$

where c_1 is a constant that can be explicitly calculated.

In a recent paper, Bae and Jung [7], improved the asymptotic formula for the second derivative of (1.3) at $s = \frac{1}{2}$, that was obtained by Andrade and Rajagopal [5], using the techniques presented by Florea. In particular, compared to the asymptotic formula obtained by Andrade and Rajagopal, Bae and Jung were able to obtain a secondary main term of size $g^3 q^{\frac{2g+1}{3}}$ whilst also bounding the error term by $q^{\frac{g}{2}(1+\epsilon)}$.

Another problem in function fields is to understand the aymptotic behaviour of

$$\sum_{D\in\mathcal{H}_{2g+2}} L(s,\chi_D)^k, \tag{1.8}$$

as $|D| \to \infty$, where $\mathcal{H}_{2g+2}$ denotes the space of monic, square-free polynomials of degree $2g+2$, which corresponds to the real quadratic function field. In particular we concentrate on when q is fixed and letting $g \to \infty$. In this context Jung [13], obtained an asymptotic formula for the first moment of (1.8) at $s = \frac{1}{2}$.

Theorem 1.3 (Jung). *Assume that q is odd and greater than 3. Then we have*

$$\sum_{D\in\mathcal{H}_{2g+2}} L\left(\frac{1}{2},\chi_D\right) = \frac{P(1)}{2\zeta_{\mathbb{A}}(2)}|D|\left[\log_q|D| + \frac{4}{\log q}\frac{P'}{P}(1) + 2\zeta_{\mathbb{A}}\left(\frac{1}{2}\right)\right] + O(|D|^{\frac{3}{4}+\frac{\log_q 2}{2}}). \tag{1.9}$$

In this paper, we will use Florea's method to improve the asymptotic formula obtained by Jung (1.9). In particular we will obtain a secondary main term of size $gq^{\frac{2g+2}{3}}$, whilst also bounding the error term by $q^{\frac{g}{2}(1+\epsilon)}$. The main result of this paper is the following Theorem.

Theorem 1.4. *Let q be a prime with $q \equiv 1 (mod\ 4)$. Then*

$$\sum_{D\in\mathcal{H}_{2g+2}} L\left(\frac{1}{2},\chi_D\right) = \frac{P(1)}{2\zeta_{\mathbb{A}}(2)}q^{2g+2}\left[(2g+2) + \frac{4}{\log q}\frac{P'}{P}(1) + 2\zeta_{\mathbb{A}}\left(\frac{1}{2}\right)\right] + q^{\frac{2g+2}{3}}\mathcal{R}(2g+2) + O(q^{\frac{g}{2}(1+\epsilon)}), \tag{1.10}$$

where $\mathcal{R}$ is a polynomial of degree 1 that can explicitly be computed (see formulas (5.35)-(5.38)).

The calculations in this paper will follow the techniques presented in Florea [9]. Using a form of the Poisson summation over $\mathbb{F}_q[x]$, Lemma 2.6, which splits the sum up into different formulas which evaluate the sums over degree f odd and degree f even, which is given in section 3. Similar

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to the imaginary function field case, the main term comes from the contribution of square polynomials f to (2.6). In section 4, we express the sum over square polynomials f as a contour integral.

In section 5, we also evaluate the non-square f using the Poisson summation formula, which we analyse the contribution from the square polynomials V. In the imaginary function field case, the contribution to the main terms come from when the degree of f is even. However, in the real function field case the contribution to the main terms from the square polynomials V come from when the degree of f is both even and odd. Thus compared to the calculations done by Florea, there are extra terms to calculate and extra integrals to evaluate. We will then show how all these terms combine with the main term to establish the asymptotic formula.

Remark 1.5. When revising this paper we came across a recent paper by Jung [15], where he computed the mean value of $L(\frac{1}{2}, \chi_D)$ when summing over all monic, square-free polynomials of degree $2g+2$ as $g \to \infty$ using similar methods to that done by Florea. Compared to Jung's paper, we explicitly go into more detail about how to calculate the asymptotic formula, especially when analysing the contribution from the square polynomials V.

2 Preliminaries and Background

We first introduce the notation which will be used throughout the article and then provide some background information on Dirichlet L-functions in function fields. We denote $\mathbb{A}^+$ to be the set of all monic polynomials in $\mathbb{F}_q[x]$ and we denote $\mathbb{A}_n^+$ and $\mathbb{A}_{\leq n}^+$ to be the set of all monic polynomials of degree n and degree at most n in $\mathbb{F}_q[x]$ respectively. Let $\mathcal{H}_n$ denote the space of monic, square-free monic polynomials over $\mathbb{F}_q[x]$ of degree n. For a polynomial $f \in \mathbb{F}_q[x]$, we denote its degree by $d(f)$ and its norm by $|f| = q^{d(f)}$. The letter P denotes a monic, irreducible polynomial over $\mathbb{F}_q[x]$.

2.1 Preliminaries on Dirichlet characters and Dirichlet L-functions for function fields

Most of the facts in this section are proved in [20]. For $\Re(s) > 1$, the zeta function of $\mathbb{A} = \mathbb{F}_q[x]$, denoted by $\zeta_{\mathbb{A}}(s)$ is defined by the infinite series

$$\zeta_{\mathbb{A}}(s) := \sum_{f \in \mathbb{A}^+} \frac{1}{|f|^s} = \prod_P (1 - |P|^{-s})^{-1}. \tag{2.1}$$

There are q^n monic polynomials of degree n, therefore we have

$$\zeta_{\mathbb{A}}(s) = (1 - q^{1-s})^{-1}.$$

We will make use of the change of variables $u = q^{-s}$, so that we write $\mathcal{Z}(u) = \zeta_{\mathbb{A}}(s)$ and thus $\mathcal{Z}(u) = (1 - qu)^{-1}$.

Assume that q is odd with $q \equiv 1 (\text{mod } 4)$. For P a monic irreducible polynomial, the quadratic residue symbol $\left(\frac{f}{P}\right) \in \{\pm 1\}$ is defined by

$$\left(\frac{f}{P}\right) \equiv f^{\frac{|P|-1}{2}} \pmod{P}$$

for f coprime to P. If $P|f$, then $\left(\frac{f}{P}\right) = 0$. We can also define the Jacobi symbol for arbitrary monic Q. Let f be coprime to Q and $Q = P_1^{e_1} \dots P_s^{e_s}$, then the Jacobi symbol is defined by

$$\left(\frac{f}{Q}\right) = \prod_{j=1}^{s} \left(\frac{f}{P_j}\right)^{e_j}.$$

Theorem 2.1 (Quadratic Reciprocity). *Let $A, B \in \mathbb{F}_q[x]$ be relatively prime and $A \neq 0$ and $B \neq 0$. Then*

$$\left(\frac{A}{B}\right) = \left(\frac{B}{A}\right)(-1)^{\frac{q-1}{2}d(A)d(B)}.$$

If we assume that $q \equiv 1(\text{mod } 4)$, then the quadratic reciprocity gives

$$\left(\frac{A}{B}\right) = \left(\frac{B}{A}\right).$$

For $g \geq 1$ we have

$$|\mathcal{H}_{2g+2}| = (q-1)q^{2g+1} = \frac{|D|}{\zeta_{\mathbb{A}}(2)}.$$

Definition 2.2. Let $D \in \mathbb{F}_q[x]$ be square-free. We define the quadratic character using the quadratic residue symbol for $\mathbb{F}_q[x]$ by

$$\chi_D(f) = \left(\frac{D}{f}\right).$$

Therefore, if $P \in \mathbb{F}_q[x]$, we have

$$\chi_D(P) = \begin{cases} 0, & \text{if } P|D, \\ 1, & \text{if } P \nmid D \text{ and } D \text{ is a square modulo } P, \\ -1, & \text{if } P \nmid D \text{ and } D \text{ is a non-square modulo } P. \end{cases}$$

Definition 2.3. The L-function corresponding to the quadratic character χ_D by

$$L(s, \chi_D) = \sum_{f \in \mathbb{A}^+} \frac{\chi_D(f)}{|f|^s},$$

which converges for $\Re(s) > 1$. For the change of variables $u = q^{-s}$, we have

$$L(s, \chi_D) = \mathcal{L}(u, \chi_D) = \sum_{f \in \mathbb{A}^+} \chi_D(f) u^{d(f)} = \prod_{P} (1 - \chi_D(f) u^{d(P)})^{-1}. \tag{2.2}$$

Since D is a monic, square-free polynomial, we have, from Proposition 4.3 in [20], that $\mathcal{L}(u, \chi_D)$ is a polynomial in u of degree $d(D) - 1$. From [22], $\mathcal{L}(u, \chi_D)$ has a trivial zero if and only if $d(D)$ is even. This enables us to define the completed L-function, $\mathcal{L}^*(u, \chi_D)$, by

$$\mathcal{L}(u, \chi_D) = (1-u)^{\lambda} \mathcal{L}^*(u, \chi_D), \tag{2.3}$$

where $\lambda = 1$ if $d(D)$ is even and $\lambda = 0$ otherwise. Then $\mathcal{L}^*(u, \chi_D)$ is a polynomial in u of degree $2\delta = d(D) - 1 - \lambda$ and satisfies the functional equation

$$\mathcal{L}^*(u, \chi_D) = (qu^2)^{\delta} \mathcal{L}^*((qu)^{-1}, \chi_D). \tag{2.4}$$

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For D a monic, square-free polynomial of degree $2g+1$ or $2g+2$, the affine equation $y^2 = D(x)$ defines a projective and connected hyperelliptic curve C_D of genus g over $\mathbb{F}_q$. The zeta function associated to C_D was first introduced by Artin [6], and is defined by

$$Z_{C_D}(u) = \exp\left(\sum_{n=1}^{\infty} N_n(C_D)\frac{u^n}{n}\right), \tag{2.5}$$

where $N_n(C_D)$ is the number of points on C_D with coordinates in a field extension $\mathbb{F}_{q^n}$ of $\mathbb{F}_q$ of degree $n \geq 1$. Weil [24], showed that

$$Z_{C_D}(u) = \frac{P_{C_D}(u)}{(1-u)(1-qu)},$$

where $P_{C_D}(u)$ is a polynomial of degree $2g$. In his thesis, Artin, proved that $P_{C_D}(u) = \mathcal{L}^*(u,\chi_D)$. Also, Weil [24], proved the Riemann Hypothesis for function fields, thus all the zeros of $\mathcal{L}^*(u,\chi_D)$ lie on the circle $|u| = q^{-\frac{1}{2}}$.

2.2 Functional Equation and Preliminary Lemmas

For $D \in \mathcal{H}_{2g+2}$, the approximate functional equation was initially proved in Jung [13], but has been corrected to match that of [21].

Lemma 2.4. *Let χ_D be a quadratic character, where $D \in \mathcal{H}_{2g+2}$. Then*

$$L\left(\frac{1}{2},\chi_D\right) = \sum_{n=0}^{g}\sum_{f\in\mathbb{A}_n^+}\chi_D(f)q^{-\frac{n}{2}} - q^{-\frac{g+1}{2}}\sum_{n=0}^{g}\sum_{f\in\mathbb{A}_n^+}\chi_D(f)$$
$$+\sum_{n=0}^{g-1}\sum_{f\in\mathbb{A}_n^+}\chi_D(f)q^{-\frac{n}{2}} - q^{-\frac{g}{2}}\sum_{n=0}^{g-1}\sum_{f\in\mathbb{A}_n^+}\chi_D(f).$$

Proof. See [13], Lemma 2.1. ∎

Using Lemma 2.4, it follows that

$$\sum_{D\in\mathcal{H}_{2g+2}} L\left(\frac{1}{2},\chi_D\right) = \sum_{f\in\mathbb{A}_{\leq g}^+}\frac{1}{\sqrt{|f|}}\sum_{D\in\mathcal{H}_{2g+2}}\chi_D(f) - q^{-\frac{g+1}{2}}\sum_{f\in\mathbb{A}_{\leq g}^+}\sum_{D\in\mathcal{H}_{2g+2}}\chi_D(f)$$
$$+\sum_{f\in\mathbb{A}_{\leq g-1}^+}\frac{1}{\sqrt{|f|}}\sum_{D\in\mathcal{H}_{2g+2}}\chi_D(f) - q^{-\frac{g}{2}}\sum_{f\in\mathbb{A}_{\leq g-1}^+}\sum_{D\in\mathcal{H}_{2g+2}}\chi_D(f). \tag{2.6}$$

We now state two Lemmas that will be used in the calculations later.

Lemma 2.5. *Let $f \in \mathbb{A}^+$. Then we have*

$$\sum_{D\in\mathcal{H}_{2g+2}}\chi_D(f) = \sum_{\substack{C|f^\infty \\ C\in\mathbb{A}_{\leq g+1}^+}}\sum_{h\in\mathbb{A}_{2g+2-2d(C)}^+}\chi_f(h) - q\sum_{\substack{C|f^\infty \\ C\in\mathbb{A}_{\leq g}^+}}\sum_{h\in\mathbb{A}_{2g-2d(C)}^+}\chi_f(h), \tag{2.7}$$

where $C|f^\infty$ means that any prime factor of C are among the prime factors of f.

Proof. The proof is similar to that given in [9], Lemma 2.2. ∎

We now state a version of Poisson summation formula over function fields. For $a \in \mathbb{F}_q((\frac{1}{t}))$, let $e(a) := e^{2\pi i a_1/q}$, where a_1 is the coefficient of $\frac{1}{t}$ in the expansion of a (for more information, see [9], section 3). For χ a general character modulo f, the generalised Gauss sum $G(V, \chi_f)$ is defined as

$$G(V,\chi) = \sum_{A \bmod f} \chi(A) e\left(\frac{AV}{f}\right). \tag{2.8}$$

The following Poisson summation formula holds.

Lemma 2.6. *Let $f \in \mathbb{A}^+$ and let m be a positive integer.*

1. *If $d(f)$ is odd, then*

$$\sum_{g \in \mathbb{A}_m^+} \chi_f(g) = \frac{q^{m+\frac{1}{2}}}{|f|} \sum_{V \in \mathbb{A}^+_{d(f)-m-1}} G(V, \chi_f). \tag{2.9}$$

2. *If $d(f)$ is even, then*

$$\sum_{g \in \mathbb{A}_m^+} \chi_f(g) = \frac{q^m}{|f|}\left(G(0,\chi_f) + (q-1)\sum_{V \in \mathbb{A}^+_{\leq d(f)-m-2}} G(V,\chi_f) - \sum_{V \in \mathbb{A}^+_{d(f)-m-1}} G(V,\chi_f)\right). \tag{2.10}$$

Proof. See [9], Proposition 3.1. ∎

Remark 2.7. $G(0, \chi_f)$ is nonzero if and only if f is a square, in which case $G(0,\chi_f) = \phi(f)$, where $\phi(f)$ is Euler's phi function for polynomials in $\mathbb{F}_q[x]$.

Lemma 2.8 (The function field analogue of Perron's formula)**.** *If the power series*

$$H(u) = \sum_{f \in \mathbb{A}^+} a(f) u^{d(f)}$$

converges absolutely for $|u| \leq R < 1$, then

$$\sum_{f \in \mathbb{A}_n^+} a(f) = \frac{1}{2\pi i} \oint_{|u|=R} \frac{H(u)}{u^{n+1}} du \tag{2.11}$$

and

$$\sum_{f \in \mathbb{A}^+_{\leq n}} a(f) = \frac{1}{2\pi i} \oint_{|u|=R} \frac{H(u)}{(1-u)u^{n+1}} du. \tag{2.12}$$

3 Setup of the Problem

Using Lemma 2.5 and the approximate functional equation (2.6), we write

$$\sum_{D \in \mathcal{H}_{2g+2}} L\left(\frac{1}{2}, \chi_D\right) = \mathcal{S}_{g,1} - \mathcal{S}_{g,2} + \mathcal{S}_{g-1,1} - \mathcal{S}_{g-1,2}, \tag{3.1}$$

where

$$\mathcal{S}_{g,1} = \sum_{f \in \mathbb{A}^+_{\leq g}} \frac{1}{\sqrt{|f|}} \sum_{\substack{C | f^\infty \\ C \in \mathbb{A}^+_{\leq g+1}}} \sum_{h \in \mathbb{A}^+_{2g+2-2d(C)}} \chi_f(h) - q \sum_{f \in \mathbb{A}^+_{\leq g}} \frac{1}{\sqrt{|f|}} \sum_{\substack{C | f^\infty \\ C \in \mathbb{A}^+_{\leq g}}} \sum_{h \in \mathbb{A}^+_{2g-2d(C)}} \chi_f(h),$$

$$\mathcal{S}_{g,2} = q^{-\frac{g+1}{2}} \sum_{f\in\mathbb{A}^+_{\le g}} \sum_{\substack{C|f^\infty \\ C\in\mathbb{A}^+_{\le g+1}}} \sum_{h\in\mathbb{A}^+_{2g+2-2d(C)}} \chi_f(h) - q^{-\frac{g-1}{2}} \sum_{f\in\mathbb{A}^+_{\le g}} \sum_{\substack{C|f^\infty \\ C\in\mathbb{A}^+_{\le g}}} \sum_{h\in\mathbb{A}^+_{2g-2d(C)}} \chi_f(h),$$

$$\mathcal{S}_{g-1,1} = \sum_{f\in\mathbb{A}^+_{\le g-1}} \frac{1}{\sqrt{|f|}} \sum_{\substack{C|f^\infty \\ C\in\mathbb{A}^+_{\le g+1}}} \sum_{h\in\mathbb{A}^+_{2g+2-2d(C)}} \chi_f(h) - q \sum_{f\in\mathbb{A}^+_{\le g-1}} \frac{1}{\sqrt{|f|}} \sum_{\substack{C|f^\infty \\ C\in\mathbb{A}^+_{\le g}}} \sum_{h\in\mathbb{A}^+_{2g-2d(C)}} \chi_f(h)$$

and

$$\mathcal{S}_{g-1,2} = q^{-\frac{g}{2}} \sum_{f\in\mathbb{A}^+_{\le g-1}} \sum_{\substack{C|f^\infty \\ C\in\mathbb{A}^+_{\le g+1}}} \sum_{h\in\mathbb{A}^+_{2g+2-2d(C)}} \chi_f(h) - q^{-\frac{g}{2}+1} \sum_{f\in\mathbb{A}^+_{\le g-1}} \sum_{\substack{C|f^\infty \\ C\in\mathbb{A}^+_{\le g}}} \sum_{h\in\mathbb{A}^+_{2g-2d(C)}} \chi_f(h).$$

From section 4 in [9], we have

$$\sum_{\substack{C|f^\infty \\ C\in\mathbb{A}^+_{g+1}}} 1 \ll q^{g\epsilon}$$

we see that the terms in $\mathcal{S}_{g,1}, \mathcal{S}_{g,2}, \mathcal{S}_{g-1,1}$ and $\mathcal{S}_{g-1,2}$ corresponding to $C \in \mathbb{A}^+_{g+1}$ are bounded by $O(q^{\frac{g}{2}(1+\epsilon)})$. Therefore, for $k \in \{g, g-1\}$, we can rewrite the terms as

$$\mathcal{S}_{k,1} = \sum_{f\in\mathbb{A}^+_{\le k}} \frac{1}{\sqrt{|f|}} \sum_{\substack{C|f^\infty \\ C\in\mathbb{A}^+_{\le g}}} \left(\sum_{h\in\mathbb{A}^+_{2g+2-2d(C)}} \chi_f(h) - q \sum_{h\in\mathbb{A}^+_{2g-2d(C)}} \chi_f(h) \right) + O(q^{\frac{g}{2}(1+\epsilon)}), \tag{3.2}$$

and

$$\mathcal{S}_{k,2} = q^{-\frac{k+1}{2}} \sum_{f\in\mathbb{A}^+_{\le k}} \sum_{\substack{C|f^\infty \\ C\in\mathbb{A}^+_{\le g}}} \left(\sum_{h\in\mathbb{A}^+_{2g+2-2d(C)}} \chi_f(h) - q \sum_{h\in\mathbb{A}^+_{2g-2d(C)}} \chi_f(h) \right) + O(q^{\frac{g}{2}(1+\epsilon)}). \tag{3.3}$$

For $\ell \in \{1, 2\}$, write

$$\mathcal{S}_{k,\ell} = \mathcal{S}^o_{k,\ell} + \mathcal{S}^e_{k,\ell} + O(q^{\frac{g}{2}(1+\epsilon)}), \tag{3.4}$$

where $\mathcal{S}^o_{k,\ell}$ and $\mathcal{S}^e_{k,\ell}$ denote the sum over $f \in \mathbb{A}^+_{\le k}$ of odd and even degree respectively. If $d(f)$ is odd, then using Lemma 2.6, we have

$$\mathcal{S}^o_{k,1} = q^{2g+\frac{5}{2}} \sum_{\substack{f\in\mathbb{A}^+_{\le k} \\ d(f) \text{ odd}}} \frac{1}{|f|} \sum_{\substack{C|f^\infty \\ C\in\mathbb{A}^+_{\le g}}} \frac{1}{|C|^2} S^o(V; f, C) \tag{3.5}$$

and

$$\mathcal{S}^o_{k,2} = q^{\frac{4g-k}{2}+2} \sum_{\substack{f\in\mathbb{A}^+_{\le k} \\ d(f) \text{ odd}}} \frac{1}{\sqrt{|f|}} \sum_{\substack{C|f^\infty \\ C\in\mathbb{A}^+_{\le g}}} \frac{1}{|C|^2} S^o(V; f, C). \tag{3.6}$$

where

$$S^o(V; f, C) = \sum_{V\in\mathbb{A}^+_{d(f)-2g-3+2d(C)}} \frac{G(V, \chi_f)}{\sqrt{|f|}} - \frac{1}{q} \sum_{V\in\mathbb{A}^+_{d(f)-2g-1+2d(C)}} \frac{G(V, \chi_f)}{\sqrt{|f|}}. \tag{3.7}$$

If $d(f)$ is even, then we rewrite $\mathcal{S}^e_{k,\ell}$ as

$$\mathcal{S}^e_{k,\ell} = M_{k,\ell} + S^e_{k,\ell,1} + S^e_{k,\ell,2}. \tag{3.8}$$

Using the remark from the previous section, we have

$$M_{k,1} = \frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)} \sum_{L \in \mathbb{A}^+_{\leq [\frac{k}{2}]}} \frac{\phi(L^2)}{|L|^3} \sum_{\substack{C | f^\infty \\ C \in \mathbb{A}^+_{\leq g}}} \frac{1}{|C|^2} \tag{3.9}$$

and

$$M_{k,2} = \frac{q^{\frac{4g-k+3}{2}}}{\zeta_{\mathbb{A}}(2)} \sum_{L \in \mathbb{A}^+_{\leq [\frac{k}{2}]}} \frac{\phi(L^2)}{|L|^2} \sum_{\substack{C | f^\infty \\ C \in \mathbb{A}^+_{\leq g}}} \frac{1}{|C|^2}. \tag{3.10}$$

Similarly, for $j \in \{1, 2\}$, we have

$$S^e_{k,1,j} = q^{2g+2} \sum_{\substack{f \in \mathbb{A}^+_{\leq k} \\ d(f) \text{ even}}} \frac{1}{|f|} \sum_{\substack{C | f^\infty \\ C \in \mathbb{A}^+_{\leq g}}} \frac{1}{|C|^2} S^e_j(V; f, C), \tag{3.11}$$

and

$$S^e_{k,2,j} = q^{\frac{4g-k+3}{2}} \sum_{\substack{f \in \mathbb{A}^+_{\leq k} \\ d(f) \text{ even}}} \frac{1}{\sqrt{|f|}} \sum_{\substack{C | f^\infty \\ C \in \mathbb{A}^+_{\leq g}}} \frac{1}{|C|^2} S^e_j(V; f, C), \tag{3.12}$$

where

$$S^e_1(V; f, C) = (q-1) \sum_{V \in \mathbb{A}^+_{\leq d(f)-2g-4+2d(C)}} \frac{G(V, \chi_f)}{\sqrt{|f|}} - \frac{q-1}{q} \sum_{V \in \mathbb{A}^+_{\leq d(f)-2g-2+2d(C)}} \frac{G(V, \chi_f)}{\sqrt{|f|}} \tag{3.13}$$

and

$$S^e_2(V; f, C) = \frac{1}{q} \sum_{V \in \mathbb{A}^+_{d(f)-2g-1+2d(C)}} \frac{G(V, \chi_f)}{\sqrt{|f|}} - \sum_{V \in \mathbb{A}^+_{d(f)-2g-3+2d(C)}} \frac{G(V, \chi_f)}{\sqrt{|f|}}. \tag{3.14}$$

Define $S^i_{k,\ell}(V = \square)$ to be the sum over V square and $S^i_{k,\ell}(V \neq \square)$ to be the sum over non-square V. Note that in Equation (3.14), when $d(f)$ is even, $d(V)$ is odd and so V cannot be a square. Also note that in Equation (3.7), when $d(f)$ is odd, $d(V)$ is even, thus there is a contribution to the main term when $d(f)$ is odd, which is not present when working in the imaginary function field case.

4 Main Term

Let

$$M = M_{g,1} - M_{g,2} + M_{g-1,1} - M_{g-1,2} \tag{4.1}$$

In this section we evaluate the main term M. The main result in this section is the following result.

Proposition 4.1. *We have that, for any $\epsilon > 0$*

$$M = M_1 + M_2 + M_3 + M_4 + O(q^{g\epsilon})$$

where

$$M_1 = \frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)} \frac{1}{2\pi i} \oint_{|u|=r} \frac{\mathcal{C}(u)}{u(1-qu)^2 (qu)^{[\frac{g}{2}]}} du,$$

$$M_2 = \frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)} \frac{1}{2\pi i} \oint_{|u|=r} \frac{\mathcal{C}(u)}{u(1-qu)^2 (qu)^{[\frac{g-1}{2}]}} du,$$

$$M_3 = -\frac{q^{\frac{3g+3}{2}}}{\zeta_{\mathbb{A}}(2)} \frac{1}{2\pi i} \oint_{|u|=r} \frac{\mathcal{C}(u)}{u(1-u)(1-qu) u^{[\frac{g}{2}]}} du$$

and

$$M_4 = -\frac{q^{\frac{3g}{2}+2}}{\zeta_{\mathbb{A}}(2)} \frac{1}{2\pi i} \oint_{|u|=r} \frac{\mathcal{C}(u)}{u(1-u)(1-qu) u^{[\frac{g-1}{2}]}} du,$$

with $r < q^{-1}$ *and*

$$\mathcal{C}(u) = \prod_P \left(1 - \frac{u^{d(P)}}{|P|+1}\right). \tag{4.2}$$

Remark 4.2. $\mathcal{C}(u)$ is analytic in $|u| < 1$. We may further write

$$\mathcal{C}(u) = \mathcal{Z}\left(\frac{u}{q}\right)^{-1} \prod_P \left(1 + \frac{u^{d(P)}}{(1+|P|)(|P| - u^{d(P)})}\right) = (1-u) \prod_P \left(1 + \frac{u^{d(P)}}{(1+|P|)(|P| - u^{d(P)})}\right) \tag{4.3}$$

which furnishes an analytic continuation of $\mathcal{C}(u)$ to the region $|u| < q$.

Proof of Proposition 4.1. From (3.9) and (3.10) and using the facts that (see [9], Section 5),

$$\sum_{\substack{C|f^\infty \\ C \in \mathbb{A}^+_{\leq g}}} \frac{1}{|C|^2} = \prod_{P|f} (1 - |P|^{-2})^{-1} + O(q^{-g(2-\epsilon)}) \text{ and } \frac{\phi(L^2)}{|L|^2} = \prod_{P|L} (1 - |P|^{-1}), \tag{4.4}$$

we have

$$M_{k,1} = \frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)} \sum_{L \in \mathbb{A}^+_{\leq [\frac{k}{2}]}} \frac{1}{|L|} \prod_{P|L} \frac{|P|}{|P|+1} + O(q^{g\epsilon}) \tag{4.5}$$

and

$$M_{k,2} = \frac{q^{\frac{4g-k+3}{2}}}{\zeta_{\mathbb{A}}(2)} \sum_{L \in \mathbb{A}^+_{\leq [\frac{k}{2}]}} \prod_{P|L} \frac{|P|}{|P|+1} + O(q^{g\epsilon}). \tag{4.6}$$

Using the function field analogue of Perron's formula (Lemma 2.8), we have

$$M_{k,1} = \frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)} \frac{1}{2\pi i} \oint_{|u|=r} \frac{\mathcal{A}(u)}{u(1-qu)(qu)^{[\frac{k}{2}]}} du \tag{4.7}$$

and

$$M_{k,2} = \frac{q^{\frac{4g-k+3}{2}}}{\zeta_{\mathbb{A}}(2)} \frac{1}{2\pi i} \oint_{|u|=r} \frac{\mathcal{A}(u)}{u(1-u) u^{[\frac{k}{2}]}} du, \tag{4.8}$$

where $r < q^{-1}$ and

$$\mathcal{A}(u) = \sum_{L \in \mathbb{A}^+} u^{d(L)} \prod_P \frac{|P|}{|P|+1}. \tag{4.9}$$

By multiplicativity, we may write

$$\mathcal{A}(u) = \prod_P \left(1 + \frac{|P|}{|P|+1} \frac{u^{d(P)}}{1 - u^{d(P)}}\right) = \mathcal{Z}(u)\mathcal{C}(u) = \frac{\mathcal{C}(u)}{(1-qu)}. \tag{4.10}$$

Inserting (4.10) into (4.7) and (4.8), we see that the integrals in the terms $M_{g,1}, M_{g,2}, M_{g-1,1}$ and $M_{g-1,2}$ are precisely the terms $M_1, M_2, -M_3$ and $-M_4$ stated in Proposition 4.1. Using (4.1), the Proposition follows. ∎

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5 Contribution from V-square

Let

$$\mathcal{S}(V=\square)=\mathcal{S}^o(V=\square)+\mathcal{S}^e(V=\square) \tag{5.1}$$

where

$$\mathcal{S}^o(V=\square)=\mathcal{S}^o_{g,1}(V=\square)-\mathcal{S}^o_{g,2}(V=\square)+\mathcal{S}^o_{g-1,1}(V=\square)-\mathcal{S}^o_{g-1,2}(V=\square) \tag{5.2}$$

and

$$S^e(V=\square)=\mathcal{S}^e_{g,1}(V=\square)-\mathcal{S}^e_{g,2}(V=\square)+\mathcal{S}^e_{g-1,1}(V=\square)-\mathcal{S}^e_{g-1,2}(V=\square). \tag{5.3}$$

In this section we will evaluate the term $\mathcal{S}(V=\square)$. The next Proposition is the main result in this section.

Proposition 5.1. *Using the same notation as before, we have that*

$$\mathcal{S}(V=\square)=\mathcal{S}_1(V=\square)+\mathcal{S}_2(V=\square)+\mathcal{S}_3(V=\square)+\mathcal{S}_4(V=\square)+q^{\frac{2g+2}{3}}\mathcal{R}(2g+2)+O(q^{\frac{g}{2}(1+\epsilon)}),$$

where

$$\mathcal{S}_1(V=\square)=-\frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)}\frac{1}{2\pi i}\oint_{|u|=R}\frac{\mathcal{C}(u)}{u(1-qu)^2(qu)^{[\frac{g}{2}]}}du,$$

$$\mathcal{S}_2(V=\square)=-\frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)}\frac{1}{2\pi i}\oint_{|u|=R}\frac{\mathcal{C}(u)}{u(1-qu)^2(qu)^{[\frac{g-1}{2}]}}du,$$

$$\mathcal{S}_3(V=\square)=\frac{q^{\frac{3g+3}{2}}}{\zeta_{\mathbb{A}}(2)}\frac{1}{2\pi i}\oint_{|u|=R}\frac{\mathcal{C}(u)}{u(1-u)(1-qu)u^{[\frac{g}{2}]}}du$$

and

$$\mathcal{S}_4(V=\square)=\frac{q^{\frac{3g}{2}+2}}{\zeta_{\mathbb{A}}(2)}\frac{1}{2\pi i}\oint_{|u|=R}\frac{\mathcal{C}(u)}{u(1-u)(1-qu)u^{[\frac{g-1}{2}]}}du,$$

with $1<R<q$ *and*

$$\mathcal{C}(u)=\prod_P\left(1-\frac{u^{d(P)}}{|P|+1}\right)=(1-u)\prod_P\left(1+\frac{u^{d(P)}}{(1+|P|)(|P|-u^{d(P)})}\right).$$

Furthermore $\mathcal{R}$ *is a linear polynomial that can be explicitly calculated, (see formulas (5.35)-(5.38)).*

Before we prove Proposition 5.1, we need the following notation and subsequent results. For $|z|>q^{-2}$, let

$$\mathcal{B}(z,w)=\sum_{f\in\mathbb{A}^+}w^{d(f)}A_f(z)\prod_{P|f}(1-|P|^{-2}z^{-d(P)})^{-1},$$

where

$$A_f(z)=\sum_{l\in\mathbb{A}^+}z^{d(l)}\frac{G(l^2,\chi_f)}{\sqrt{|f|}}.$$

Then we have the following results.

Lemma 5.2. *For $|z| > q^{-2}$, we have*

$$\mathcal{B}(z,w) = \mathcal{Z}(z)\mathcal{Z}(w)\mathcal{Z}(qw^2z)\prod_P \mathcal{B}_P(z,w), \tag{5.4}$$

where

$$\mathcal{B}_P(z,w) = 1+\frac{w^{d(P)} - (zw^2)^{d(P)}|P|^2 - (z^2w)^{d(P)}|P|^2 + (z^2w^3)^{d(P)}|P|^2 + (zw^2)^{d(P)}|P| - (zw^3)^{d(P)}|P|}{z^{d(P)}|P|^2 - 1}.$$

Moreover $\prod_P \mathcal{B}_P(z,w)$ converges absolutely for $|w| < q|z|, |w| < q^{-\frac{1}{2}}$ and $|wz| < q^{-1}$.

Proof. See [9], Lemma 6.2. ∎

Lemma 5.3. *We have*

$$\prod_P \mathcal{B}_P(z,w) = \mathcal{Z}\left(\frac{w}{q^2z}\right)\mathcal{Z}(w^2)^{-1}\prod_P \mathcal{D}_P(z,w), \tag{5.5}$$

where

$$\mathcal{D}_P(z,w)$$
$$= 1 + \frac{-w^{2d(P)} - \frac{w^{3d(P)}}{|P|} + \frac{w^{d(P)}}{z^{d(P)}|P|^2} + (zw^2)^{d(P)}|P| + (zw^2)^{d(P)} - (z^2w)^{d(P)}|P|^2 + (zw^3)^{d(P)} - (z^2w^2)^{d(P)}|P|^2}{(z^{d(P)}|P|^2 - 1)(1 + w^{d(P)})}.$$

Moreover $\mathcal{D}_P(z,w)$ converges absolutely for $|w|^2 < q|z|, |w| < q^3|z|^2, |w| < 1$ and $|wz| < q^{-1}$.

Proof. See [9], Lemma 6.3 ∎

Outline of the Proof of Proposition 5.1: From the Poisson summation formula the sum over square polynomials V will occur when the degree of f is even and when the degree of f is odd. In the next two subsections, we will find two integrals for each $\mathcal{S}^i_{k,\ell}(V = \square)$ corresponding to simple poles at $w = q^{-1}$ and $w = qz$. In the third subsection we will manipulate the integrals corresponding to the pole at $w = q^{-1}$, similar to that done in section 6 [9], which will yield the main terms. In the final subsection, we will evaluate the integrals corresponding to the pole at $w = qz$, which will yield the secondary main term.

5.1 Degree f even

In this subsection, we prove the following result.

Lemma 5.4. *We have*

$$\mathcal{S}^e(V = \square) = \mathcal{A}^e_{g,1} - \mathcal{A}^e_{g,2} + \mathcal{A}^e_{g-1,1} - \mathcal{A}^e_{g-1,2} + \mathcal{B}^e_{g,1} - \mathcal{B}^e_{g,2} + \mathcal{B}^e_{g-1,1} - \mathcal{B}^e_{g-1,2} + O(q^{\frac{g}{2}(1+\epsilon)}), \tag{5.6}$$

where $\mathcal{A}^e_{k,\ell}$ and $\mathcal{B}^e_{k,\ell}$ are the integrals stated at the end of the subsection.

Proof. From (3.13) and using the function field analogue of Perron's formula, we obtain

$$S^e_1(l^2; f, C) = \frac{1}{2\pi i}\oint_{|z|=q^{-1-\epsilon}} \frac{z^g(q-1)(qz-1)A_f(z)}{q(1-z)z^{\frac{d(f)}{2}+d(C)}}dz.$$

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Also, using the fact that (see [9], Proof of Lemma 6.1),

$$\sum_{\substack{C|f^\infty \\ C\in\mathbb{A}_{\le g}^+}} \frac{1}{|C|^2 z^{d(C)}} = \prod_{P|f}(1-z^{-d(P)}|P|^{-2})^{-1} + O(q^{g(\epsilon-1)}),$$

we have, for $k\in\{g,g-1\}$,

$$\mathcal{S}_{k,1}^e(V=\square) = \frac{q^{2g+2}}{2\pi i}\oint_{|z|=q^{-1-\epsilon}} \frac{z^g(q-1)(qz-1)}{q(1-z)} H_{k,1}^e(z)dz + O(q^{\frac{g}{2}(1+\epsilon)}),$$

where

$$H_{k,1}^e(z) = \sum_{\substack{f\in\mathbb{A}_{\le k}^+ \\ d(f)\text{ even}}} \frac{A_f(z)}{|f|z^{\frac{d(f)}{2}}}\prod_{P|f}(1-|P|^{-2}z^{-d(P)})^{-1}.$$

Similarly we have

$$\mathcal{S}_{k,2}^e(V=\square) = \frac{q^{\frac{4g-k+3}{2}}}{2\pi i}\oint_{|z|=q^{-1-\epsilon}} \frac{z^g(q-1)(qz-1)}{q(1-z)} H_{k,2}^e(z)dz + O(q^{\frac{g}{2}(1+\epsilon)}),$$

where

$$H_{k,2}^e(z) = \sum_{\substack{f\in\mathbb{A}_{\le k}^+ \\ d(f)\text{ even}}} \frac{A_f(z)}{\sqrt{|f|}z^{\frac{d(f)}{2}}}\prod_{P|f}(1-|P|^{-2}z^{-d(P)})^{-1}.$$

Using the function field analogue of Perron's formula, we have for $\ell\in\{1,2\}$

$$H_{k,\ell}^e(z) = \frac{1}{2\pi i}\oint_{|w|=r_2} \frac{\mathcal{B}(z,w)}{w(1-q^{3-\ell}zw^2)(q^{3-\ell}zw^2)^{[\frac{k}{2}]}}dw - \frac{1}{2\pi i}\oint_{|w|=r_2}\frac{q^{3-\ell}zw\mathcal{B}(z,w)}{1-q^{3-\ell}zw^2}dw. \tag{5.7}$$

For each $k\in\{g,g-1\}$ and $\ell\in\{1,2\}$, the second integral in (5.7), is zero since the integrand has no poles inside the circle $|w|=r_2<q^{-1}$. Therefore we have

$$\mathcal{S}_{k,1}^e(V=\square) = \frac{q^{2g+2}}{(2\pi i)^2}\oint_{|z|=q^{-1-\epsilon}}\oint_{|w|=r_2} \frac{z^g(q-1)(qz-1)\mathcal{B}(z,w)}{qw(1-z)(1-q^2zw^2)(q^2zw^2)^{[\frac{k}{2}]}}dwdz + O(q^{\frac{g}{2}(1+\epsilon)})$$

and

$$\mathcal{S}_{k,2}^e(V=\square) = \frac{q^{\frac{4g-k+3}{2}}}{(2\pi i)^2}\oint_{|z|=q^{-1-\epsilon}}\oint_{|w|=r_2} \frac{z^g(q-1)(qz-1)\mathcal{B}(z,w)}{qw(1-z)(1-qzw^2)(qzw^2)^{[\frac{k}{2}]}}dwdz + O(q^{\frac{g}{2}(1+\epsilon)}).$$

Using equation (5.4) in Lemma 5.2 we obtain

$$\mathcal{S}_{k,1}^e(V=\square) = -\frac{q^{2g+2}}{(2\pi i)^2}\oint_{|z|=q^{-1-\epsilon}}\oint_{|w|=r_2} \frac{z^g(q-1)\prod_P\mathcal{B}_P(z,w)}{qw(1-z)(1-qw)(1-q^2zw^2)^2(q^2zw^2)^{[\frac{k}{2}]}}dwdz + O(q^{\frac{g}{2}(1+\epsilon)})$$

and

$$\mathcal{S}_{k,2}^e(V=\square) = -\frac{q^{\frac{4g-k+3}{2}}}{(2\pi i)^2}\oint_{|z|=q^{-1-\epsilon}}\oint_{|w|=r_2} \frac{z^g(q-1)\prod_P\mathcal{B}_P(z,w)}{qw(1-z)(1-qw)(1-qzw^2)(1-q^2zw^2)(qzw^2)^{[\frac{k}{2}]}}dwdz + O(q^{\frac{g}{2}(1+\epsilon)}).$$

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Using equation (5.5) in Lemma 5.3, we obtain

$$\mathcal{S}_{k,1}^e(V=\square) = -\frac{q^{2g+2}}{(2\pi i)^2}\oint_{|z|=q^{-1-\epsilon}}\oint_{|w|=r_2}\frac{z^g(q-1)(1-qw^2)\prod_P\mathcal{D}_P(z,w)}{qw(1-z)(1-qw)(1-\frac{w}{qz})(1-q^2zw^2)^2(q^2zw^2)^{[\frac{k}{2}]}}dwdz$$
$$+O(q^{\frac{g}{2}(1+\epsilon)})$$

and

$$\mathcal{S}_{k,2}^e(V=\square) = -\frac{q^{\frac{4g-k+3}{2}}}{(2\pi i)^2}\oint_{|z|=q^{-1-\epsilon}}\oint_{|w|=r_2}\frac{z^g(q-1)(1-qw^2)\prod_P\mathcal{D}_P(z,w)}{qw(1-z)(1-qw)(1-\frac{w}{qz})(1-qzw^2)(1-q^2zw^2)(qzw^2)^{[\frac{k}{2}]}}dwdz$$
$$+O(q^{\frac{g}{2}(1+\epsilon)}).$$

Shrinking the contour $|z|=q^{-1-\epsilon}$ to $|z|=q^{-\frac{3}{2}}$, we do not encounter any poles. Enlarging the contour $|w|=r_2<q^{-1}$ to $|w|=q^{-\frac{1}{4}-\epsilon}$, we encounter two simple poles, one at $w=q^{-1}$ and one at $w=qz$. Evaluating the residues at $w=q^{-1}$ and $w=qz$ and writing

$$\mathcal{S}_{k,\ell}^e(V=\square)=\mathcal{A}_{k,\ell}^e+\mathcal{B}_{k,\ell}^e+\mathcal{C}_{k,\ell}^e+O(q^{\frac{g}{2}(1+\epsilon)}), \tag{5.8}$$

whilst using Lemma 6.3 where we have for each k,ℓ, $\mathcal{C}_{k,\ell}^e\ll q^{\frac{g}{2}(1+\epsilon)}$, then we have

$$\mathcal{A}_{g,1}^e=-\frac{q^{2g+2}}{2\pi i}\oint_{|z|=q^{-\frac{3}{2}}}\frac{z^g(q-1)\prod_P\mathcal{B}_P(z,q^{-1})}{q(1-z)^3z^{[\frac{g}{2}]}}dz,$$
$$\mathcal{B}_{g,1}^e=-\frac{q^{2g+2}}{2\pi i}\oint_{|z|=q^{-\frac{3}{2}}}\frac{z^g(q-1)(1-q^3z^2)\prod_P\mathcal{D}_P(z,qz)}{q(1-z)(1-q^2z)(1-q^4z^3)^2(q^4z^3)^{[\frac{g}{2}]}}dz,$$

$$\mathcal{A}_{g,2}^e=-\frac{q^{\frac{3g+3}{2}}}{2\pi i}\oint_{|z|=q^{-\frac{3}{2}}}\frac{z^g(q-1)\prod_P\mathcal{B}_P(z,q^{-1})}{q(1-z)^2(1-q^{-1}z)(q^{-1}z)^{[\frac{g}{2}]}}dz,$$
$$\mathcal{B}_{g,2}^e=-\frac{q^{\frac{3g+3}{2}}}{2\pi i}\oint_{|z|=q^{-\frac{3}{2}}}\frac{z^g(q-1)(1-q^3z^2)\prod_P\mathcal{D}_P(z,qz)}{q(1-z)(1-q^2z)(1-q^3z^3)(1-q^4z^3)(q^3z^3)^{[\frac{g}{2}]}}dz,$$

$$\mathcal{A}_{g-1,1}^e=-\frac{q^{2g+2}}{2\pi i}\oint_{|z|=q^{-\frac{3}{2}}}\frac{z^g(q-1)\prod_P\mathcal{B}_P(z,q^{-1})}{q(1-z)^3z^{[\frac{g-1}{2}]}}dz,$$
$$\mathcal{B}_{g-1,1}^e=-\frac{q^{2g+2}}{2\pi i}\oint_{|z|=q^{-\frac{3}{2}}}\frac{z^g(q-1)(1-q^3z^2)\prod_P\mathcal{D}_P(z,qz)}{q(1-z)(1-q^2z)(1-q^4z^3)^2(q^4z^3)^{[\frac{g-1}{2}]}}dz$$

and

$$\mathcal{A}_{g-1,2}^e=-\frac{q^{\frac{3g}{2}+2}}{2\pi i}\oint_{|z|=q^{-\frac{3}{2}}}\frac{z^g(q-1)\prod_P\mathcal{B}_P(z,q^{-1})}{q(1-z)^2(1-q^{-1}z)(q^{-1}z)^{[\frac{g-1}{2}]}}dz,$$
$$\mathcal{B}_{g-1,2}^e=\frac{q^{\frac{3g}{2}+2}}{2\pi i}\oint_{|z|=q^{-\frac{3}{2}}}\frac{z^g(q-1)(1-q^3z^2)\prod_P\mathcal{D}_P(z,qz)}{q(1-z)(1-q^2z)(1-q^3z^3)(1-q^4z^3)(q^3z^3)^{[\frac{g-1}{2}]}}dz.$$

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5.2 Degree f odd

In this subsection, we prove the following result.

Lemma 5.5. *We have*

$$\mathcal{S}^o(V=\square)=\mathcal{A}^o_{g,1}-\mathcal{A}^o_{g,2}+\mathcal{A}^o_{g-1,1}-\mathcal{A}^o_{g-1,2}+\mathcal{B}^o_{g,1}-\mathcal{B}^o_{g,2}+\mathcal{B}^o_{g-1,1}-\mathcal{B}^o_{g-1,2}+O(q^{\frac{g}{2}(1+\epsilon)}), \tag{5.9}$$

where $\mathcal{A}^o_{k,\ell}$ *and* $\mathcal{B}^o_{k,\ell}$ *are the integrals stated at the end of the subsection.*

Proof. From (3.7) and using the function field analogue of Perron's formula, we have

$$S^o(l^2;f,C)=\frac{1}{2\pi i}\oint_{|z|=q^{-1-\epsilon}}\frac{A_f(z)z^{g-\frac{1}{2}}(qz-1)}{qz^{\frac{d(f)}{2}+d(C)}}dz.$$

Also, using the fact that (see, [9], Proof of Lemma 6.1),

$$\sum_{\substack{C|f^\infty\\ C\in\mathbb{A}^+_{\leq g}}}\frac{1}{|C|^2z^{d(C)}}=\prod_{P|f}(1-z^{-d(P)}|P|^{-2})^{-1}+O(q^{g(\epsilon-1)}),$$

we have

$$\mathcal{S}^o_{k,1}(V=\square)=\frac{q^{2g+\frac{5}{2}}}{2\pi i}\oint_{|z|=q^{-1-\epsilon}}\frac{z^{g-\frac{1}{2}}(qz-1)}{q}H^o_{k,1}(z)dz+O(q^{\frac{g}{2}(1+\epsilon)}),$$

where

$$H^o_{k,1}(z)=\sum_{\substack{f\in\mathbb{A}^+_{\leq k}\\ d(f)\text{ odd}}}\frac{A_f(z)}{|f|z^{\frac{d(f)}{2}}}\prod_{P|f}(1-|P|^{-2}z^{-d(P)})^{-1}.$$

Similarly we have

$$\mathcal{S}^o_{k,2}(V=\square)=\frac{q^{\frac{4g-k}{2}+2}}{2\pi i}\oint_{|z|=q^{-1-\epsilon}}\frac{z^{g-\frac{1}{2}}(qz-1)}{q}H^o_{k,2}(z)dz+O(q^{\frac{g}{2}(1+\epsilon)}),$$

where

$$H^o_{k,2}(z)=\sum_{\substack{f\in\mathbb{A}^+_{\leq k}\\ d(f)\text{ odd}}}\frac{A_f(z)}{\sqrt{|f|}z^{\frac{d(f)}{2}}}\prod_{P|f}(1-|P|^{-2}z^{-d(P)})^{-1}.$$

Using the function field analogue of Perron's formula, we have for $\ell\in\{1,2\}$,

$$H^o_{g,\ell}(z)=\frac{1}{2\pi i}\oint_{|w|=r_2}\frac{\mathcal{B}(z,w)}{q^{\frac{3-\ell}{2}}z^{\frac{1}{2}}w^2(1-q^{3-\ell}zw^2)(q^{3-\ell}zw^2)^{[\frac{g-1}{2}]}}dw-\frac{1}{2\pi i}\oint_{|w|=r_2}\frac{q^{\frac{3-\ell}{2}}z^{\frac{1}{2}}\mathcal{B}(z,w)}{1-q^{3-\ell}zw^2}dw \tag{5.10}$$

and

$$H^o_{g-1,\ell}(z)=\frac{1}{2\pi i}\oint_{|w|=r_2}\frac{q^{\frac{3-\ell}{2}}z^{\frac{1}{2}}\mathcal{B}(z,w)}{(1-q^{3-\ell}zw^2)(q^{3-\ell}zw^2)^{[\frac{g}{2}]}}dw-\frac{1}{2\pi i}\oint_{|w|=r_2}\frac{q^{\frac{3-\ell}{2}}z^{\frac{1}{2}}\mathcal{B}(z,w)}{1-q^{3-\ell}zw^2}dw. \tag{5.11}$$

For each $\ell\in\{1,2\}$, the second integrals in (5.10) and (5.11) are zero since the integrands have no poles inside the circle $|w|=r_2<q^{-1}$. Therefore we have

$$\mathcal{S}^o_{k,1}(V=\square)=\frac{q^{2g+\frac{5}{2}}}{(2\pi i)^2}\oint_{|z|=q^{-1-\epsilon}}\oint_{|w|=r_2}\frac{z^{k-(-1)^{g-k}}(qz-1)\mathcal{B}(z,w)}{q^{2(k-g+1)}w^{2(k-g+1)}(1-q^2zw^2)(q^2zw^2)^{\left[\frac{k-(-1)^{g-k}}{2}\right]}}dwdz+O(q^{\frac{g}{2}(1+\epsilon)})$$

and

$$\mathcal{S}_{k,2}^o(V=\square) = \frac{q^{\frac{4g-k}{2}+2}}{(2\pi i)^2}\oint_{|z|=q^{-1-\epsilon}}\oint_{|w|=r_2}\frac{q^{g-k-\frac{3}{2}}w^{-2(k-g+1)}z^{k-(-1)^{g-k}}(qz-1)\mathcal{B}(z,w)}{(1-qzw^2)(qzw^2)^{\left[\frac{k-(-1)^{g-k}}{2}\right]}}dwdz + O(q^{\frac{g}{2}(1+\epsilon)}).$$

Using equation (5.4) in Lemma 5.2 we have

$$\mathcal{S}_{k,1}^o(V=\square) = -\frac{q^{2g+\frac{5}{2}}}{(2\pi i)^2}\oint_{|z|=q^{-1-\epsilon}}\oint_{|w|=r_2}\frac{z^{k-(-1)^{g-k}}\prod_P\mathcal{B}_P(z,w)}{q^{2(k-g+1)}w^{2(k-g+1)}(1-qw)(1-q^2zw^2)(q^2zw^2)^{\left[\frac{k-(-1)^{g-k}}{2}\right]}}dwdz$$
$$+O(q^{\frac{g}{2}(1+\epsilon)})$$

and

$$\mathcal{S}_{k,2}^o(V=\square) = -\frac{q^{\frac{4g-k}{2}+2}}{(2\pi i)^2}\oint_{|z|=q^{-1-\epsilon}}\oint_{|w|=r_2}\frac{q^{g-k-\frac{3}{2}}w^{-2(k-g+1)}z^{k-(-1)^{g-k}}\prod_P\mathcal{B}_P(z,w)}{(1-qw)(1-qzw^2)(1-q^2zw^2)(qzw^2)^{\left[\frac{k-(-1)^{g-k}}{2}\right]}}dwdz$$
$$+O(q^{\frac{g}{2}(1+\epsilon)}).$$

Using equation (5.5) in Lemma 5.3, we have

$$\mathcal{S}_{k,1}^o(V=\square) = -\frac{q^{2g+\frac{5}{2}}}{(2\pi i)^2}\oint_{|z|=q^{-1-\epsilon}}\oint_{|w|=r_2}\frac{z^{k-(-1)^{g-k}}(1-qw^2)\prod_P\mathcal{D}_P(z,w)}{q^{2(k-g+1)}w^{2(k-g+1)}(1-qw)(1-\frac{w}{qz})(1-q^2zw^2)(q^2zw^2)^{\left[\frac{k-(-1)^{g-k}}{2}\right]}}dwdz$$
$$+O(q^{\frac{g}{2}(1+\epsilon)})$$

and

$$\mathcal{S}_{k,2}^o(V=\square) = -\frac{q^{\frac{4g-k}{2}+2}}{(2\pi i)^2}\oint_{|z|=q^{-1-\epsilon}}\oint_{|w|=r_2}\frac{q^{g-k-\frac{3}{2}}w^{-2(k-g+1)}z^{k-(-1)^{g-k}}(1-qw^2)\prod_P\mathcal{D}_P(z,w)}{(1-qw)(1-\frac{w}{qz})(1-qzw^2)(1-q^2zw^2)(qzw^2)^{\left[\frac{k-(-1)^{g-k}}{2}\right]}}dwdz$$
$$+O(q^{\frac{g}{2}(1+\epsilon)}).$$

Shrinking the contour $|z|=q^{-1-\epsilon}$ to $|z|=q^{-\frac{3}{2}}$, we do not encounter any poles. Enlarging the contour $|w|=r_2<q^{-1}$ to $|w|=q^{-\frac{1}{4}-\epsilon}$, we encounter two simple poles, one at $w=q^{-1}$ and one at $w=qz$. Evaluating the residues at $w=q^{-1}$ and $w=qz$ and writing

$$\mathcal{S}_{k,\ell}^o(V=\square) = \mathcal{A}_{k,\ell}^o+\mathcal{B}_{k,\ell}^o+\mathcal{C}_{k,\ell}^o+O(q^{\frac{g}{2}(1+\epsilon)}), \tag{5.12}$$

whilst using Lemma 6.3 where we have for each k,ℓ, $\mathcal{C}_{k,\ell}^o \ll q^{\frac{g}{2}(1+\epsilon)}$, then we have

$$\mathcal{A}_{g,1}^o = -\frac{q^{2g+\frac{5}{2}}}{2\pi i}\oint_{|z|=q^{-\frac{3}{2}}}\frac{z^{g-1}\prod_P\mathcal{B}_P(z,q^{-1})}{q(1-z)^2z^{\left[\frac{g-1}{2}\right]}}dz,$$
$$\mathcal{B}_{g,1}^o = -\frac{q^{2g+\frac{5}{2}}}{2\pi i}\oint_{|z|=q^{-\frac{3}{2}}}\frac{z^{g-2}(1-q^3z^2)\prod_P\mathcal{D}_P(z,qz)}{q^3(1-q^2z)(1-q^4z^3)^2(q^4z^3)^{\left[\frac{g-1}{2}\right]}}dz,$$

$$\mathcal{A}^{o}_{g,2} = -\frac{q^{\frac{3g}{2}+2}}{2\pi i}\oint_{|z|=q^{-\frac{3}{2}}} \frac{z^{g-1}\prod_P \mathcal{B}_P(z,q^{-1})}{q^{\frac{1}{2}}(1-z)(1-q^{-1}z)(q^{-1}z)^{[\frac{g-1}{2}]}}dz,$$

$$\mathcal{B}^{o}_{g,2} = -\frac{q^{\frac{3g}{2}+2}}{2\pi i}\oint_{|z|=q^{-\frac{3}{2}}} \frac{z^{g-2}(1-q^3z^2)\prod_P \mathcal{D}_P(z,qz)}{q^{\frac{5}{2}}(1-q^2z)(1-q^3z^3)(1-q^4z^3)(q^3z^3)^{[\frac{g-1}{2}]}}dz,$$

$$\mathcal{A}^{o}_{g-1,1} = -\frac{q^{2g+\frac{5}{2}}}{2\pi i}\oint_{|z|=q^{-\frac{3}{2}}} \frac{z^{g}\prod_P \mathcal{B}_P(z,q^{-1})}{q(1-z)^2z^{[\frac{g}{2}]}}dz,$$

$$\mathcal{B}^{o}_{g-1,1} = -\frac{q^{2g+\frac{5}{2}}}{2\pi i}\oint_{|z|=q^{-\frac{3}{2}}} \frac{qz^{g+1}(1-q^3z^2)\prod_P \mathcal{D}_P(z,qz)}{(1-q^2z)(1-q^4z^3)^2(q^4z^3)^{[\frac{g}{2}]}}dz$$

and

$$\mathcal{A}^{o}_{g-1,2} = -\frac{q^{\frac{3g+5}{2}}}{2\pi i}\oint_{|z|=q^{-\frac{3}{2}}} \frac{z^{g}\prod_P \mathcal{B}_P(z,q^{-1})}{q^{\frac{3}{2}}(1-z)(1-q^{-1}z)(q^{-1}z)^{[\frac{g}{2}]}}dz,$$

$$\mathcal{B}^{o}_{g-1,2} = -\frac{q^{\frac{3g+5}{2}}}{2\pi i}\oint_{|z|=q^{-\frac{3}{2}}} \frac{q^{\frac{1}{2}}z^{g+1}(1-q^3z^2)\prod_P \mathcal{D}_P(z,qz)}{(1-q^2z)(1-q^3z^3)(1-q^4z^3)(q^3z^3)^{[\frac{g}{2}]}}dz.$$

■

5.3 Contribution from $\mathcal{A}$ Terms

In this subsection, we will focus on evaluating the $\mathcal{A}$ terms which corresponds to the pole at $w = q^{-1}$, these will give the main terms in Proposition 5.1. Let

$$\mathcal{A} = \mathcal{A}^e_{g,1} - \mathcal{A}^e_{g,2} + \mathcal{A}^e_{g-1,1} - \mathcal{A}^e_{g-1,2} + \mathcal{A}^o_{g,1} - \mathcal{A}^o_{g,2} + \mathcal{A}^o_{g-1,1} - \mathcal{A}^o_{g-1,2},$$

then, the main result in this subsection is the following.

Lemma 5.6. *Using the same notation as before, we have*

$$\mathcal{A} = \mathcal{S}_1(V=\Box) + \mathcal{S}_2(V=\Box) + \mathcal{S}_3(V=\Box) + \mathcal{S}_4(V=\Box),$$

where, in particular, the terms $\mathcal{S}_1(V=\Box), \mathcal{S}_2(V=\Box), \mathcal{S}_3(V=\Box)$ and $\mathcal{S}_4(V=\Box)$ are the integrals stated in Proposition 5.1.

Proof. For each $k \in \{g, g-1\}$, rewrite $\mathcal{A}^e_{k,1}$ as

$$\mathcal{A}^e_{k,1} = -\frac{q^{2g+2}}{2\pi i}\oint_{|z|=q^{-\frac{3}{2}}} \frac{z^g(q-\frac{1}{z}+\frac{1}{z}-1)\prod_P \mathcal{B}_P(z,q^{-1})}{q(1-z)^3z^{[\frac{k}{2}]}}dz.$$

Let

$$\mathcal{A}^e_{k,1} = \mathcal{A}^e_{k,1,1} + \mathcal{A}^e_{k,1,2}$$

where

$$\mathcal{A}^e_{k,1,1} = -\frac{q^{2g+2}}{2\pi i}\oint_{|z|=q^{-\frac{3}{2}}} \frac{z^g(1-\frac{1}{qz})\prod_P \mathcal{B}_P(z,q^{-1})}{(1-z)^3z^{[\frac{k}{2}]}}dz$$

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and

$$\mathcal{A}^e_{k,1,2} = -\frac{q^{2g+2}}{2\pi i}\oint_{|z|=q^{-\frac{3}{2}}}\frac{z^{g-1}\prod_P \mathcal{B}_P(z,q^{-1})}{q(1-z)^2 z^{[\frac{k}{2}]}}dz$$

Using the change of variables $z=(qu)^{-1}$, the contour of integration becomes the circle around the origin $|u|=\sqrt{q}$ and note that (from Lemma 5.2) $\prod_P \mathcal{B}_P(\frac{1}{qu},\frac{1}{q})$ is absolutely convergent for $q^{-1}<|u|<q$. Thus, we have

$$\mathcal{A}^e_{k,1,1} = -\frac{q^{2g+2}}{2\pi i}\oint_{|u|=\sqrt{q}}\frac{(1-u)\prod_P \mathcal{B}_P\left(\frac{1}{qu},\frac{1}{q}\right)\left(1-\frac{1}{qu}\right)^{-1}}{u(1-qu)^2(qu)^{\left[\frac{k-(-1)^{g-k}}{2}\right]}}du.$$

Using the fact (see [9], section 6) that

$$(1-u)\prod_P \mathcal{B}_P\left(\frac{1}{qu},\frac{1}{q}\right)\left(1-\frac{1}{qu}\right)^{-1} = \frac{\mathcal{C}(u)}{\zeta_{\mathbb{A}}(2)}, \tag{5.13}$$

we get that

$$\mathcal{A}^e_{g,1,1} = -\frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)}\frac{1}{2\pi i}\oint_{|u|=\sqrt{q}}\frac{\mathcal{C}(u)}{u(1-qu)^2(qu)^{[\frac{g-1}{2}]}}du \tag{5.14}$$

and

$$\mathcal{A}^e_{g-1,1,1} = -\frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)}\frac{1}{2\pi i}\oint_{|u|=\sqrt{q}}\frac{\mathcal{C}(u)}{u(1-qu)^2(qu)^{[\frac{g}{2}]}}du. \tag{5.15}$$

We see that that (5.14) and (5.15) are precisely the terms $\mathcal{S}_1(V=\square)$ and $\mathcal{S}_2(V=\square)$ given in the statement of Lemma 5.6. Similarly, using the substitution $z=(qu)^{-1}$ we have

$$\mathcal{A}^e_{k,1,2} = -\frac{q^{2g+2}}{2\pi i}\oint_{|u|=\sqrt{q}}\frac{\prod_P \mathcal{B}_P\left(\frac{1}{qu},\frac{1}{q}\right)}{(1-qu)^2(qu)^{\left[\frac{k-(-1)^{g-k}}{2}\right]}}du$$

and

$$\mathcal{A}^o_{k,1} = -\frac{q^{2g+\frac{5}{2}}}{2\pi i}\oint_{|u|=\sqrt{q}}\frac{\prod_P \mathcal{B}_P\left(\frac{1}{qu},\frac{1}{q}\right)}{(qu)^{g-k}(1-qu)^2(qu)^{[\frac{k}{2}]}}du$$

Using a variant of (5.13) we have

$$\mathcal{A}^e_{k,1,2} = \frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)}\frac{1}{2\pi i}\oint_{|u|=\sqrt{q}}\frac{\mathcal{C}(u)}{qu(1-u)(1-qu)(qu)^{\left[\frac{k-(-1)^{g-k}}{2}\right]}}du, \tag{5.16}$$

and

$$\mathcal{A}^o_{k,1} = \frac{q^{2g+\frac{5}{2}}}{\zeta_{\mathbb{A}}(2)}\frac{1}{2\pi i}\oint_{|u|=\sqrt{q}}\frac{\mathcal{C}(u)}{(qu)^{g-k+1}(1-u)(1-qu)(qu)^{[\frac{k}{2}]}}du. \tag{5.17}$$

Rewrite $\mathcal{A}^o_{g-1,1}$ as

$$\mathcal{A}^o_{g-1} = \frac{q^{2g+\frac{5}{2}}}{\zeta_{\mathbb{A}}(2)}\frac{1}{2\pi i}\oint_{|u|=\sqrt{q}}\frac{\mathcal{C}(u)(1-qu+qu)}{q^2u^2(1-u)(1-qu)(qu)^{[\frac{g-1}{2}]}}du. \tag{5.18}$$

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Then, we let

$$\mathcal{A}^o_{g-1,1} = \mathcal{A}^o_{g-1,1,1} + \mathcal{A}^o_{g-1,1,2},$$

where

$$\mathcal{A}^o_{g-1,1,1} = \frac{q^{2g+\frac{5}{2}}}{\zeta_{\mathbb{A}}(2)}\frac{1}{2\pi i}\oint_{|u|=\sqrt{q}}\frac{\mathcal{C}(u)}{qu(1-u)(1-qu)(qu)^{\left[\frac{g-1}{2}\right]}}du.$$

and

$$\mathcal{A}^o_{g-1,1,2} = \frac{q^{2g+\frac{5}{2}}}{\zeta_{\mathbb{A}}(2)}\frac{1}{2\pi i}\oint_{|u|=\sqrt{q}}\frac{\mathcal{C}(u)}{(1-u)(qu)^{\left[\frac{g-1}{2}\right]+2}}du.$$

Combining $\mathcal{A}^o_{g,1}$ and $\mathcal{A}^e_{g-1,1,2}$, we have

$$\mathcal{A}^o_{g,1} + \mathcal{A}^e_{g-1,1,2} = \frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)}\frac{1}{2\pi i}\oint_{|u|=\sqrt{q}}\frac{\mathcal{C}(u)(1+q^{\frac{1}{2}})}{qu(1-u)(1-qu)(qu)^{\left[\frac{g}{2}\right]}}du.$$

Using the fact that (see [13], Proof of Main Theorem)

$$1 + q^{\frac{1}{2}} = q^{-\frac{g-1}{2}+\left[\frac{g}{2}\right]} + q^{-\frac{g}{2}+\left[\frac{g-1}{2}\right]+1}, \tag{5.19}$$

we have

$$\mathcal{A}^o_{g,1} + \mathcal{A}^e_{g-1,1,2} = \frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)}\frac{1}{2\pi i}\oint_{|u|=\sqrt{q}}\frac{\mathcal{C}(u)(q^{-\frac{g-1}{2}+\left[\frac{g}{2}\right]} + q^{-\frac{g}{2}+\left[\frac{g-1}{2}\right]+1})}{qu(1-u)(1-qu)(qu)^{\left[\frac{g}{2}\right]}}du.$$

Let

$$\mathcal{A}^o_{g,1} + \mathcal{A}^e_{g-1,1,2} = \hat{\mathcal{A}}_1 + \hat{\mathcal{A}}_2,$$

where

$$\hat{\mathcal{A}}_1 = \frac{q^{\frac{3g+3}{2}}}{\zeta_{\mathbb{A}}(2)}\frac{1}{2\pi i}\oint_{|u|=\sqrt{q}}\frac{\mathcal{C}(u)}{u(1-u)(1-qu)u^{\left[\frac{g}{2}\right]}}du \tag{5.20}$$

and

$$\hat{\mathcal{A}}_2 = \frac{q^{\frac{3g}{2}+3+\left[\frac{g-1}{2}\right]}}{\zeta_{\mathbb{A}}(2)}\frac{1}{2\pi i}\oint_{|u|=\sqrt{q}}\frac{\mathcal{C}(u)}{(1-u)(1-qu)(qu)^{\left[\frac{g}{2}\right]+1}}du.$$

Similarly combining $\mathcal{A}^o_{g-1,1,1}$ and $\mathcal{A}^e_{g,1,2}$ and using (5.19), we have

$$\mathcal{A}^o_{g-1,1,1} + \mathcal{A}^e_{g,1,2} = \tilde{\mathcal{A}}_1 + \tilde{\mathcal{A}}_2,$$

where

$$\tilde{\mathcal{A}}_1 = \frac{q^{\frac{3g}{2}+2}}{\zeta_{\mathbb{A}}(2)}\frac{1}{2\pi i}\oint_{|u|=\sqrt{q}}\frac{\mathcal{C}(u)}{u(1-u)(1-qu)u^{\left[\frac{g-1}{2}\right]}}du \tag{5.21}$$

and

$$\tilde{\mathcal{A}}_2 = \frac{q^{\frac{3g+5}{2}+\left[\frac{g}{2}\right]}}{\zeta_{\mathbb{A}}(2)}\frac{1}{2\pi i}\oint_{|u|=\sqrt{q}}\frac{\mathcal{C}(u)}{(1-u)(1-qu)(qu)^{\left[\frac{g-1}{2}\right]+1}}du. \tag{5.22}$$

We see that that $\hat{\mathcal{A}}_1$ and $\tilde{\mathcal{A}}_1$ are precisely the terms $\mathcal{S}_3(V=\square)$ and $\mathcal{S}_4(V=\square)$ given in the statement of Lemma 5.6. From (4.3), we have that $\mathcal{C}(1)=0$, thus, inside the circle $|u|=\sqrt{q}$, $\mathcal{A}^o_{g-1,1,2}$ has a pole of order $\left[\frac{g-1}{2}\right]+2$ at $u=0$. Using the Residue Theorem we have that

$$\mathcal{A}^o_{g-1,1,2} = \frac{q^{2g+\frac{1}{2}-\left[\frac{g-1}{2}\right]}}{\zeta_{\mathbb{A}}(2)}\sum_{n=0}^{\left[\frac{g-1}{2}\right]+1}\frac{\mathcal{C}^{(n)}(0)}{n!}. \tag{5.23}$$

Similarly, inside the circle $|u| = \sqrt{q}$, the integrals $\hat{\mathcal{A}}_2$ and $\tilde{\mathcal{A}}_2$ have a simple pole at $u = q^{-1}$ and a pole at $u = 0$ of order $\left[\frac{g}{2}\right] + 1$ and $\left[\frac{g-1}{2}\right] + 1$ respectively. Thus we have

$$\hat{\mathcal{A}}_2 = \frac{q^{\frac{5g}{2}+1-2\left[\frac{g}{2}\right]}}{\zeta_{\mathbb{A}}(2)} \sum_{n=0}^{\left[\frac{g}{2}\right]} \frac{\mathcal{C}^{(n)}(0)}{n!} \sum_{k=0}^{\left[\frac{g}{2}\right]-n} q^k - \frac{q^{\frac{3g}{2}+3+\left[\frac{g-1}{2}\right]}}{\zeta_{\mathbb{A}}(2)(q-1)}$$

and

$$\tilde{\mathcal{A}}_2 = \frac{q^{\frac{5g+1}{2}-2\left[\frac{g-1}{2}\right]}}{\zeta_{\mathbb{A}}(2)} \sum_{n=0}^{\left[\frac{g-1}{2}\right]} \frac{\mathcal{C}^{(n)}(0)}{n!} \sum_{k=0}^{\left[\frac{g-1}{2}\right]-n} q^k - \frac{q^{\frac{3g+5}{2}+\left[\frac{g}{2}\right]}}{\zeta_{\mathbb{A}}(2)(q-1)}.$$

For the remaining integrals, we rewrite $\mathcal{A}_{k,2}^{e}$ as

$$\mathcal{A}_{k,2}^{e} = \mathcal{A}_{k,2,1}^{e} + \mathcal{A}_{k,2,2}^{e}$$

where

$$\mathcal{A}_{k,2,1}^{e} = -\frac{q^{\frac{4g-k+3}{2}}}{2\pi i} \oint_{|z|=q^{-\frac{3}{2}}} \frac{z^g(1-\frac{1}{qz})\prod_P \mathcal{B}_P(z,q^{-1})}{(1-z)^2(1-q^{-1}z)(q^{-1}z)^{\left[\frac{k}{2}\right]}} dz$$

and

$$\mathcal{A}_{k,2,2}^{e} = -\frac{q^{\frac{4g-k+3}{2}}}{2\pi i} \oint_{|z|=q^{-\frac{3}{2}}} \frac{z^{g-1}\prod_P \mathcal{B}_P(z,q^{-1})}{q(1-z)(1-q^{-1}z)(q^{-1}z)^{\left[\frac{k}{2}\right]}} dz.$$

Using the substitution $z = (qu)^{-1}$ we have

$$\mathcal{A}_{k,2,1}^{e} = \frac{q^{\frac{6g-k+5}{2}}}{2\pi i} \oint_{|u|=\sqrt{q}} \frac{(1-u)\prod_P \mathcal{B}_P\left(\frac{1}{qu},\frac{1}{q}\right)}{(1-qu)^2(1-q^2u)(q^2u)^{\left[\frac{k-(-1)^{g-k}}{2}\right]}} du,$$

$$\mathcal{A}_{k,2,2}^{e} = -\frac{q^{\frac{6g-k+3}{2}}}{2\pi i} \oint_{|u|=\sqrt{q}} \frac{\prod_P \mathcal{B}_P\left(\frac{1}{qu},\frac{1}{q}\right)}{(1-qu)(1-q^2u)(q^2u)^{\left[\frac{k-(-1)^{g-k}}{2}\right]}} du$$

and

$$\mathcal{A}_{k,2}^{o} = -\frac{q^{\frac{6g-k+5}{2}}}{2\pi i} \oint_{|u|=\sqrt{q}} \frac{\prod_P \mathcal{B}_P\left(\frac{1}{qu},\frac{1}{q}\right)}{(q^2u)^{g-k}(1-qu)(1-q^2u)(q^2u)^{\left[\frac{k}{2}\right]}} du.$$

Using a variant of (5.13), we have

$$\mathcal{A}_{k,2,1}^{e} = -\frac{q^{\frac{6g-k+7}{2}}}{\zeta_{\mathbb{A}}(2)} \frac{1}{2\pi i} \oint_{|u|=\sqrt{q}} \frac{\mathcal{C}(u)}{(1-qu)(1-q^2u)(q^2u)^{\left[\frac{k-(-1)^{g-k}}{2}\right]+1}} du, \tag{5.24}$$

$$\mathcal{A}_{k,2,2}^{e} = \frac{q^{\frac{6g-k+5}{2}}}{\zeta_{\mathbb{A}}(2)} \frac{1}{2\pi i} \oint_{|u|=\sqrt{q}} \frac{\mathcal{C}(u)}{(1-u)(1-q^2u)(q^2u)^{\left[\frac{k-(-1)^{g-k}}{2}\right]+1}} du, \tag{5.25}$$

and

$$\mathcal{A}_{k,2}^{o} = \frac{q^{\frac{6g-k+7}{2}}}{\zeta_{\mathbb{A}}(2)} \frac{1}{2\pi i} \oint_{|u|=\sqrt{q}} \frac{\mathcal{C}(u)}{(q^2u)^{g-k}(1-u)(1-q^2u)(q^2u)^{\left[\frac{k}{2}\right]+1}} du. \tag{5.26}$$

Inside the circle $|u| = \sqrt{q}$, the integrals all have poles at $u = 0, u = q^{-1}$ and $u = q^{-2}$ of varying orders, thus using the Residue Theorem, we have that

$$\mathcal{A}^e_{g,2,1} = -\frac{q^{\frac{5g+3}{2}-2[\frac{g-1}{2}]}}{\zeta_{\mathbb{A}}(2)} \sum_{n=0}^{[\frac{g-1}{2}]} \frac{\mathcal{C}^{(n)}(0)}{n!} \sum_{k=([\frac{g-1}{2}]-n)}^{2([\frac{g-1}{2}]-n)} q^k - \frac{q^{\frac{5g+3}{2}-[\frac{g-1}{2}]}}{\zeta_{\mathbb{A}}(2)} \frac{\mathcal{C}(q^{-1})}{q-1} + \frac{q^{\frac{5g+5}{2}}}{\zeta_{\mathbb{A}}(2)} \frac{\mathcal{C}(q^{-2})}{q-1}, \tag{5.27}$$

$$\mathcal{A}^e_{g,2,2} = \frac{q^{\frac{5g+1}{2}-2[\frac{g-1}{2}]}}{\zeta_{\mathbb{A}}(2)} \sum_{n=0}^{[\frac{g-1}{2}]} \frac{\mathcal{C}^{(n)}(0)}{n!} \sum_{k=0}^{[\frac{g-1}{2}]-n} q^{2k} - \frac{q^{\frac{5g+5}{2}}}{\zeta_{\mathbb{A}}(2)} \frac{\mathcal{C}(q^{-2})}{q^2-1}, \tag{5.28}$$

$$\mathcal{A}^e_{g-1,2,1} = -\frac{q^{\frac{5g}{2}+2-2[\frac{g}{2}]}}{\zeta_{\mathbb{A}}(2)} \sum_{n=0}^{[\frac{g}{2}]} \frac{\mathcal{C}^{(n)}(0)}{n!} \sum_{k=([\frac{g}{2}]-n)}^{2([\frac{g}{2}]-n)} q^k - \frac{q^{\frac{5g}{2}+2-[\frac{g}{2}]}}{\zeta_{\mathbb{A}}(2)} \frac{\mathcal{C}(q^{-1})}{q-1} + \frac{q^{\frac{5g}{2}+3}}{\zeta_{\mathbb{A}}(2)} \frac{\mathcal{C}(q^{-2})}{q-1}, \tag{5.29}$$

$$\mathcal{A}^e_{g-1,2,2} = \frac{q^{\frac{5g}{2}+1-2[\frac{g}{2}]}}{\zeta_{\mathbb{A}}(2)} \sum_{n=0}^{[\frac{g}{2}]} \frac{\mathcal{C}^{(n)}(0)}{n!} \sum_{k=0}^{[\frac{g}{2}]-n} q^{2k} - \frac{q^{\frac{5g}{2}+3}}{\zeta_{\mathbb{A}}(2)} \frac{\mathcal{C}(q^{-2})}{q^2-1}, \tag{5.30}$$

$$\mathcal{A}^o_{g,2} = \frac{q^{\frac{5g+3}{2}-2[\frac{g}{2}]}}{\zeta_{\mathbb{A}}(2)} \sum_{n=0}^{[\frac{g}{2}]} \frac{\mathcal{C}^{(n)}(0)}{n!} \sum_{k=0}^{[\frac{g}{2}]-n} q^{2k} - \frac{q^{\frac{5g+7}{2}}}{\zeta_{\mathbb{A}}(2)} \frac{\mathcal{C}(q^{-2})}{q^2-1} \tag{5.31}$$

and

$$\mathcal{A}^o_{g-1,2} = \frac{q^{\frac{5g}{2}-2[\frac{g-1}{2}]}}{\zeta_{\mathbb{A}}(2)} \sum_{n=0}^{[\frac{g-1}{2}]+1} \frac{\mathcal{C}^{(n)}(0)}{n!} \sum_{k=0}^{[\frac{g-1}{2}]+1-n} q^{2k} - \frac{q^{\frac{5g}{2}+4}}{\zeta_{\mathbb{A}}(2)} \frac{\mathcal{C}(q^{-2})}{q^2-1}. \tag{5.32}$$

To complete the proof, we want to show that

$$\mathcal{A}^o_{g-1,1,2} + \hat{A}_2 + \tilde{A}_2 - \mathcal{A}^e_{g,2,1} - \mathcal{A}^e_{g,2,2} - \mathcal{A}^e_{g-1,2,1} - \mathcal{A}^e_{g-1,2,2} - \mathcal{A}^o_{g,2} - \mathcal{A}^o_{g-1,2}, \tag{5.33}$$

equals zero. Using the fact that (see [14], section 1)

$$q^{g-[\frac{g-1}{2}]} - q^{[\frac{g}{2}]+1} = 0 \text{ and } q^{g-[\frac{g}{2}]} - q^{[\frac{g-1}{2}]+1} = 0, \tag{5.34}$$

we see that the terms corresponding to residue at $u = q^{-1}$ and $u = q^{-2}$ equal zero. Finally, we use induction on g (see appendix) to show that the terms corresponding to the residue at $u = 0$ equals zero. Thus (5.33) equals zero. ■

5.4 Contribution from $\mathcal{B}$ Terms

We will now focus on evaluating the $\mathcal{B}$ terms which corresponds to the pole at $w = qz$, these will give the secondary main term in Proposition 5.1. Let

$$\mathcal{B} = \mathcal{B}^e_{g,1} - \mathcal{B}^e_{g,2} + \mathcal{B}^e_{g-1,1} - \mathcal{B}^e_{g-1,2} + \mathcal{B}^o_{g,1} - \mathcal{B}^o_{g,2} + \mathcal{B}^o_{g-1,1} - \mathcal{B}^o_{g-1,2},$$

then, the main result in this subsection is the following.

Lemma 5.7. *Using the same notation as before, we have that*

$$\mathcal{B} = q^{\frac{2g+2}{3}} \mathcal{R}(2g+2) + O(q^{\frac{g}{2}(1+\epsilon)}),$$

where $\mathcal{R}$ is a polynomial of degree 1 which can be explicitly calculated (see formulas (5.35)-(5.38)).

Proof. For each $j \in \{o, e\}$ and $k \in \{g, g-1\}$, we write

$$\mathcal{B}_{k,1}^j = -\frac{q^{2g+2}}{2\pi i} \oint_{|z|=q^{-\frac{3}{2}}} F_{k,1}^j(z) dz \quad \text{and} \quad \mathcal{B}_{k,2}^j = -\frac{q^{\frac{3g}{2}+2}}{2\pi i} \oint_{|z|=q^{-\frac{3}{2}}} F_{k,2}^j(z) dz.$$

Enlarging the contour $|z| = q^{-\frac{3}{2}}$ to $|z| = q^{-1-\epsilon}$ we encounter a double pole at $z = q^{-\frac{4}{3}}$ of $F_{k,1}^j(z)$ and a simple pole at $z = q^{-\frac{4}{3}}$ of $F_{k,2}^j(z)$. From Lemma 5.3, $\prod_P \mathcal{D}_P(z, qz)$ is absolutely convergent when $q^{-2} < |z| < q^{-1}$. Then, we have

$$\mathcal{B}_{k,1}^j = q^{2g+2} \text{Res}(F_{k,1}^j(z); z = q^{-\frac{4}{3}}) - \frac{q^{2g+2}}{2\pi i} \oint_{|z|=q^{-1-\epsilon}} F_{k,1}^j(z) dz$$

and

$$\mathcal{B}_{k,2}^j = q^{\frac{3g}{2}+2} \text{Res}(F_{k,2}^j(z); z = q^{-\frac{4}{3}}) - \frac{q^{\frac{3g}{2}+2}}{2\pi i} \oint_{|z|=q^{-1-\epsilon}} F_{k,2}^j(z) dz$$

where the second terms are bounded by $O(q^{\frac{g}{2}(1+\epsilon)})$. Then

$$\mathcal{B}_{g,1}^e = q^{\frac{2g+2}{3}} R_1(g) + O(q^{\frac{g}{2}(1+\epsilon)}), \qquad \mathcal{B}_{g-1,1}^e = q^{\frac{2g+2}{3}} R_2(g) + O(q^{\frac{g}{2}(1+\epsilon)}),$$
$$\mathcal{B}_{g,1}^o = q^{\frac{2g+2}{3}} R_3(g) + O(q^{\frac{g}{2}(1+\epsilon)}) \text{ and } \mathcal{B}_{g-1,1}^o = q^{\frac{2g+2}{3}} R_4(g) + O(q^{\frac{g}{2}(1+\epsilon)}).$$

where each R_i is a linear polynomial whose coefficients can be computed explicitly. Let

$$q^{\frac{2g+2}{3}} \mathcal{R}_1(2g+2) = q^{\frac{2g+2}{3}} R_1(g) + q^{\frac{2g+2}{3}} R_2(g) + q^{\frac{2g+2}{3}} R_3(g) + q^{\frac{2g+2}{3}} R_4(g),$$

where

$$\mathcal{R}_1(x) = \frac{\zeta_{\mathbb{A}}\left(\frac{5}{3}\right) \zeta_{\mathbb{A}}\left(\frac{7}{3}\right)}{9 q^{\frac{4}{3}} \zeta_{\mathbb{A}}\left(\frac{4}{3}\right)} \prod_P \mathcal{D}_P(q^{-\frac{4}{3}}, q^{-\frac{1}{3}}) \left[\frac{x}{2} C_3 - C_4 - \frac{2C_3}{q^{\frac{4}{3}}} \frac{\frac{d}{dz} \prod_P \mathcal{D}_P(z, qz)}{\prod_P \mathcal{D}_P(z, qz)} \Big|_{z=q^{-\frac{4}{3}}} \right], \tag{5.35}$$

$$C_3 = 1 - q - q^{\frac{7}{6}} + q^{-\frac{1}{6}}$$

and

$$C_4 = 4C_3 \zeta_{\mathbb{A}}\left(\frac{4}{3}\right) - C_3 \zeta_{\mathbb{A}}\left(\frac{5}{3}\right) + \frac{2(q-1)\zeta_{\mathbb{A}}\left(\frac{7}{3}\right)}{q^{\frac{4}{3}}} + 4(q-1) + 2q^{\frac{1}{6}} \zeta_{\mathbb{A}}\left(\frac{7}{3}\right)(1+q).$$

Also we write

$$\mathcal{B}_{g,2}^e = -q^{\frac{g}{6}+\left[\frac{g}{2}\right]} C_1^e + O(q^{\frac{g}{2}(1+\epsilon)}), \qquad \mathcal{B}_{g-1,2}^e = -q^{\frac{g}{6}+\left[\frac{g-1}{2}\right]} C_2^e + O(q^{\frac{g}{2}(1+\epsilon)}),$$
$$\mathcal{B}_{g,2}^o = -q^{\frac{g}{6}+\left[\frac{g-1}{2}\right]} C_1^o + O(q^{\frac{g}{2}(1+\epsilon)}) \text{ and } \mathcal{B}_{g-1,2}^o = -q^{\frac{g}{6}+\left[\frac{g}{2}\right]} C_2^o + O(q^{\frac{g}{2}(1+\epsilon)}).$$

where each C_ℓ^j are constants that can be explicitly computed. Let

$$C_1 = C_1^e + C_2^o \qquad \text{and} \qquad C_2 = C_1^o + C_2^e,$$

then we have

$$C_1 = \frac{\zeta_{\mathbb{A}}\left(\frac{5}{3}\right) \zeta_{\mathbb{A}}\left(\frac{7}{3}\right) \zeta_{\mathbb{A}}(2)}{\zeta_{\mathbb{A}}\left(\frac{4}{3}\right)} (q^{-\frac{1}{6}} - q^{-\frac{7}{6}} + q^{-\frac{4}{3}} - 1) \prod_P \mathcal{D}_P(q^{-\frac{4}{3}}, q^{-\frac{1}{3}}) \tag{5.36}$$

and

$$C_2 = \frac{\zeta_{\mathbb{A}}\left(\frac{5}{3}\right) \zeta_{\mathbb{A}}\left(\frac{7}{3}\right) \zeta_{\mathbb{A}}(2)}{\zeta_{\mathbb{A}}\left(\frac{4}{3}\right)} (q^{\frac{1}{3}} - q^{-\frac{2}{3}} + q^{\frac{11}{6}} - q) \prod_P \mathcal{D}_P(q^{-\frac{4}{3}}, q^{-\frac{1}{3}}). \tag{5.37}$$

Moreover,

$$\prod_P \mathcal{D}_P(q^{-\frac{4}{3}}, q^{-\frac{1}{3}}) = \prod_P \left(1 - \frac{|P|^{\frac{4}{3}} + |P|^{\frac{2}{3}} + |P|^{\frac{1}{3}} + 1}{(|P|^{\frac{4}{3}} + |P|)^2}\right)$$

and

$$\frac{1}{q^{\frac{4}{3}}} \frac{\frac{d}{dz}\prod_P \mathcal{D}_P(z, qz)}{\prod_P \mathcal{D}_P(z, qz)}\Big|_{|z|=q^{-\frac{4}{3}}} = -\sum_P \frac{d(P)(|P|-1)(|P|^{\frac{1}{3}}+1)}{(|P|^{\frac{1}{3}}-1)(|P|^{\frac{4}{3}}+|P|)^2}.$$

Letting

$$q^{\frac{2g+2}{3}}\mathcal{R}(2g+2) = q^{\frac{2g+2}{3}}\mathcal{R}_1(2g+2) + C_1 q^{\frac{g}{6}+[\frac{g}{2}]} + C_2 q^{\frac{g}{6}+[\frac{g-1}{2}]} \tag{5.38}$$

proves the Lemma. ∎

Proposition 5.1 is immediate from Lemma 5.4, Lemma 5.5, Lemma 5.6, Lemma 5.7 and (5.1).

6 Error From Non-Square V

Let

$$\mathcal{S}(V \neq \square) = \mathcal{S}^o(V \neq \square) + \mathcal{S}^e(V \neq \square), \tag{6.1}$$

where

$$\mathcal{S}^o(V \neq \square) = \mathcal{S}^o_{g,1}(V \neq \square) - \mathcal{S}^o_{g,2}(V \neq \square) + \mathcal{S}^o_{g-1,1}(V \neq \square) - \mathcal{S}^o_{g-1,2}(V \neq \square) \tag{6.2}$$

and

$$\mathcal{S}^e(V \neq \square) = \mathcal{S}^e_{g,1}(V \neq \square) - \mathcal{S}^e_{g,2}(V \neq \square) + \mathcal{S}^e_{g-1,1}(V \neq \square) - \mathcal{S}^e_{g-1,2}(V \neq \square). \tag{6.3}$$

Then, in this section we will bound the term $\mathcal{S}(V \neq \square)$. The next Proposition is the main result in this section.

Proposition 6.1. *Using the notation described previously, we have, for any $\epsilon > 0$,*

$$\mathcal{S}(V \neq \square) \ll q^{\frac{g}{2}(1+\epsilon)}. \tag{6.4}$$

To prove Proposition 6.1, we will need the following results (see [9], section 7). We have

$$\sum_{\substack{C|f^\infty \\ C\in\mathbb{A}_m^+}} \frac{1}{|C|^2} = \frac{1}{2\pi i}\oint_{|u|=r_1} \frac{1}{q^{2m}u^{m+1}\prod_{P|f}(1-u^{d(P)})} du \tag{6.5}$$

with $r_1 < 1$. For a non-square $V \in \mathbb{A}^+$ and positive integer n, let

$$\delta_{V;n}(u) = \sum_{f\in\mathbb{A}_n^+} \frac{G(V,\chi_f)}{\sqrt{|f|}\prod_{P|f}(1-u^{d(P)})}.$$

Then, if $|u| = q^{-\epsilon}$, then we have

$$|\delta_{V;n}(u)| \ll q^{\frac{n}{2}(1+\epsilon)}. \tag{6.6}$$

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6.1 Bounding $\mathcal{S}^e(V \neq \square)$

For each $k \in \{g, g-1\}$ and $\ell \in \{1,2\}$, we have

$$\mathcal{S}^e_{k,\ell}(V \neq \square) = \mathcal{S}^e_{k,\ell,1}(V \neq \square) + \mathcal{S}^e_{k,\ell,2}(V \neq \square).$$

Write

$$\mathcal{S}^e_{k,\ell,1}(V \neq \square) = \tilde{\mathcal{S}}^e_{k,\ell,1}(V \neq \square) - \hat{\mathcal{S}}^e_{k,\ell,1}(V \neq \square)$$

and

$$\mathcal{S}^e_{k,\ell,2}(V \neq \square) = \tilde{\mathcal{S}}^e_{k,\ell,2}(V \neq \square) - \hat{\mathcal{S}}^e_{k,\ell,2}(V \neq \square),$$

where $\tilde{\mathcal{S}}^e_{k,\ell,1}(V \neq \square)$, and $\hat{\mathcal{S}}^e_{k,\ell,1}(V \neq \square)$ denote the sum over non-square V of degree $d(V) \leq d(f) - 2g - 4 + 2d(C)$ and $d(V) \leq d(f) - 2g - 2 + 2d(C)$ respectively. Similarly $\tilde{\mathcal{S}}^e_{k,\ell,2}(V \neq \square)$ and $\hat{\mathcal{S}}^e_{k,\ell,2}(V \neq \square)$ denote the sum over V with $d(V) = d(f)-2g-1+2d(C)$ and $d(V) = d(f)-2g-3+2d(C)$ respectively. Then, by (6.5), we can write

$$\tilde{\mathcal{S}}^e_{g,1,1}(V \neq \square) = \frac{(q-1)q^{2g+2}}{2\pi i} \oint_{|u|=r_1} \sum_{n=0}^{[\frac{g}{2}]} \frac{1}{q^{2k}} \sum_{m=g-n+2}^{g} \frac{1}{q^{2m}u^{m+1}} \sum_{\square \neq V \in \mathbb{A}^+_{\leq 2n-2g-4+2d(C)}} \delta_{V;2n}(u) du,$$

$$\hat{\mathcal{S}}^e_{g,1,1}(V \neq \square) = \frac{(q-1)q^{2g+1}}{2\pi i} \oint_{|u|=r_1} \sum_{n=0}^{[\frac{g}{2}]} \frac{1}{q^{2k}} \sum_{m=g-n+1}^{g} \frac{1}{q^{2m}u^{m+1}} \sum_{\square \neq V \in \mathbb{A}^+_{\leq 2n-2g-2+2d(C)}} \delta_{V;2n}(u) du$$

and

$$\tilde{\mathcal{S}}^e_{g,1,2}(V \neq \square) = \frac{q^{2g+1}}{2\pi i} \oint_{|u|=r_1} \sum_{n=0}^{[\frac{g}{2}]} \frac{1}{q^{2k}} \sum_{m=g-n+1}^{g} \frac{1}{q^{2m}u^{m+1}} \sum_{V \in \mathbb{A}^+_{2n-2g-1+2m}} \delta_{V;2n}(u) du,$$

$$\hat{\mathcal{S}}^e_{g,1,2}(V \neq \square) = \frac{q^{2g+2}}{2\pi i} \oint_{|u|=r_1} \sum_{n=0}^{[\frac{g}{2}]} \frac{1}{q^{2k}} \sum_{m=g-n+2}^{g} \frac{1}{q^{2m}u^{m+1}} \sum_{V \in \mathbb{A}^+_{2n-2g-3+2m}} \delta_{V;2n}(u) du,$$

with $r_1 < 1$. Using (6.6), we can bound $\delta_{V;2n}(u)$ and trivially bounding the sum over V, we get that $\tilde{\mathcal{S}}^e_{g,1,1}(V \neq \square) \ll q^{\frac{g}{2}(1+\epsilon)}$, $\hat{\mathcal{S}}^e_{g,1,1}(V \neq \square) \ll q^{\frac{g}{2}(1+\epsilon)}$, $\tilde{\mathcal{S}}^e_{g,1,2}(V \neq \square) \ll q^{\frac{g}{2}(1+\epsilon)}$ and $\hat{\mathcal{S}}^e_{g,1,2}(V \neq \square) \ll q^{\frac{g}{2}(1+\epsilon)}$, thus $\mathcal{S}^e_{g,1}(V \neq \square) \ll q^{\frac{g}{2}(1+\epsilon)}$. Using the same calculations, we can bound $\mathcal{S}^e_{g,2}(V \neq \square), \mathcal{S}^e_{g-1,1}(V \neq \square)$ and $\mathcal{S}^e_{g-1,2}(V \neq \square)$ by $q^{\frac{g}{2}(1+\epsilon)}$, hence $\mathcal{S}^e(V \neq \square) \ll q^{\frac{g}{2}(1+\epsilon)}$.

6.2 Bounding $\mathcal{S}^o(V \neq \square)$

For $k \in \{g, g-1\}, \ell \in \{1,2\}$, let

$$\mathcal{S}^o_{k,\ell}(V \neq \square) = \tilde{\mathcal{S}}^o_{k,\ell}(V \neq \square) - \hat{\mathcal{S}}^o_{k,\ell}(V \neq \square),$$

where $\tilde{\mathcal{S}}^o_{k,\ell}$ and $\hat{\mathcal{S}}^o_{k,\ell}$ denotes the sum over non-square V with $d(V) = d(f) - 2g - 3 + 2d(C)$ and $d(V) = d(f) - 2g - 1 + 2d(C)$ respectively. Then, using (6.5), we have

$$\tilde{\mathcal{S}}^o_{g,1}(V \neq \square) = \frac{q^{2g+\frac{5}{2}}}{2\pi i} \oint_{|u|=r_1} \sum_{n=0}^{[\frac{g-1}{2}]} \frac{1}{q^{2n+1}} \sum_{m=g-n+1}^{g} \frac{1}{q^{2m}u^{m+1}} \sum_{\square \neq V \in \mathbb{A}^+_{2n-2g-2+2m}} \delta_{V;2n+1}(u) du$$

and

$$\hat{\mathcal{S}}^o_{g,1}(V\neq\square)=\frac{q^{2g+\frac{3}{2}}}{2\pi i}\oint_{|u|=r_1}\sum_{n=0}^{[\frac{g-1}{2}]}\frac{1}{q^{2n+1}}\sum_{m=g-n}^{g}\frac{1}{q^{2m}u^{m+1}}\sum_{\square\neq V\in\mathbb{A}^+_{2n-2g+2m}}\delta_{V;2n+1}(u)du$$

with $r_1<1$. Using (6.6) to bound $\delta_{V;2n+1}(u)$ and trivially bounding the sum over V, we get that $\hat{\mathcal{S}}^o_{g,1}(V\neq\square)\ll q^{\frac{g}{2}(1+\epsilon)}$ and $\tilde{\mathcal{S}}^o_{g,1}(V\neq\square)\ll q^{\frac{g}{2}(1+\epsilon)}$. Thus $\mathcal{S}^o_{g,1}(V\neq\square)\ll q^{\frac{g}{2}(1+\epsilon)}$. Similar calculations can be used to bound $\mathcal{S}^o_{g,2}(V\neq\square)$, $\mathcal{S}^o_{g-1,1}(V\neq\square)$ and $\mathcal{S}^o_{g-1,2}(V\neq\square)$ by $q^{\frac{g}{2}(1+\epsilon)}$. Therefore $\mathcal{S}^o(V\neq\square)\ll q^{\frac{g}{2}(1+\epsilon)}$, which proves Proposition 6.1.

7 Proof of Theorem 1.4

We combine the results from the previous sections to prove Theorem 1.4.

Proof of Theorem 1.4. Using (3.1), we have

$$\sum_{D\in\mathcal{H}_{2g+2}}L\left(\frac{1}{2},\chi_D\right)=\mathcal{S}_{g,1}-\mathcal{S}_{g,2}+\mathcal{S}_{g-1,1}-\mathcal{S}_{g-1,2}. \tag{7.1}$$

Using equations in previous sections, we can rewrite (7.1) as

$$\sum_{D\in\mathcal{H}_{2g+2}}L\left(\frac{1}{2},\chi_D\right)=M+S(V=\square)+S(V\neq\square). \tag{7.2}$$

Using Proposition 4.1, Proposition 5.1 and Proposition 6.1, we have

$$\sum_{D\in\mathcal{H}_{2g+2}}L\left(\frac{1}{2},\chi_D\right)=M_1+M_2+M_3+M_4+\mathcal{S}_1(V=\square)+\mathcal{S}_2(V=\square)+\mathcal{S}_3(V=\square)+\mathcal{S}_4(V=\square)$$
$$+q^{\frac{2g+2}{3}}\mathcal{R}(2g+2)+O(q^{\frac{g}{2}(1+\epsilon)}).$$

By Remark 4.2, $\mathcal{C}(u)$ has an analytic continuation for $|u|<q$ and $\mathcal{C}(1)=0$, therefore between the circles $|u|=r$ and $|u|=R$, the integrands corresponding to the terms $M_1,M_2,\mathcal{S}_1(V=\square)$ and $\mathcal{S}_2(V=\square)$ have a double pole at $u=q^{-1}$. Similarly the integrands corresponding to the terms $M_3,M_4,\mathcal{S}_3(V=\square)$ and $\mathcal{S}_4(V=\square)$ have a simple pole at $u=q^{-1}$. Computing the residue at $u=q^{-1}$, we get that

$$\mathcal{S}_1(V=\square)+M_1=\frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)}P(1)\left(\left[\frac{g}{2}\right]+1+\frac{1}{\log q}\frac{P'(1)}{P(1)}\right),$$

$$\mathcal{S}_2(V=\square)+M_2=\frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)}P(1)\left(\left[\frac{g-1}{2}\right]+1+\frac{1}{\log q}\frac{P'(1)}{P(1)}\right),$$

$$\mathcal{S}_3(V=\square)+M_3=\frac{q^{\frac{3g+5}{2}+[\frac{g}{2}]}}{\zeta_{\mathbb{A}}(2)}\frac{P(1)}{(q-1)}$$

and

$$\mathcal{S}_4(V=\square)+M_4=\frac{q^{\frac{3g}{2}+3+[\frac{g-1}{2}]}}{\zeta_{\mathbb{A}}(2)}\frac{P(1)}{(q-1)}.$$

where $\mathcal{C}(u)=P(s)$ with the change of variables $u=q^{-s}$. Putting everything together and using equation (5.19) the Theorem follows. ∎

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A Completing the Proof of Lemma 5.6

In the appendix we prove the claim that

$$\mathcal{A}^o_{g-1,1,2} + \hat{\mathcal{A}}_2 + \tilde{\mathcal{A}}_2 - \mathcal{A}^e_{g,2,1} - \mathcal{A}^e_{g,2,2} - \mathcal{A}^e_{g-1,2,1} - \mathcal{A}^e_{g-1,2,2} - \mathcal{A}^o_{g,2} - \mathcal{A}^o_{g-1,2} \tag{A.1}$$

equals zero. For the terms corresponding to the residues at $u = q^{-1}$ and $u = q^{-2}$ we have shown that (A.1) equals zero, thus it remains to show that for the terms corresponding to the residue at $u = 0$, (A.1) equals zero. We prove this using induction on g. To do this we consider two cases, the first when g is even and second when g is odd.

A.1 g even

Let $g = 2m$ for $m \in \mathbb{Z}$, then we will show that $(A.1)$ equals zero for all $m \geq 1$. For the base case, $m = 1$, we have that (A.1) is equal to

$$\begin{aligned}\frac{1}{\zeta_{\mathbb{A}}(2)}\Bigg(& q^{\frac{9}{2}}\big(\mathcal{C}(0) + \mathcal{C}'(0)\big) + q^4\big(\mathcal{C}(0)(1+q) + \mathcal{C}'(0)\big) + q^{\frac{11}{2}}\mathcal{C}(0) + q^{\frac{13}{2}}\mathcal{C}(0) - q^{\frac{11}{2}}\mathcal{C}(0)\\ &+ q^5\big(\mathcal{C}(0)(q+q^2) + \mathcal{C}'(0)\big) - q^4\big(\mathcal{C}(0)(1+q^2) + \mathcal{C}'(0)\big) - q^{\frac{9}{2}}\big(\mathcal{C}(0)(1+q^2) + \mathcal{C}'(0)\big)\\ &- q^5\big(\mathcal{C}(0)(1+q^2) + \mathcal{C}'(0)\big)\Bigg),\end{aligned} \tag{A.2}$$

which when cancelling the terms equals zero, hence the base case is true. Assume that (A.1)= 0, for $m = t$. Then we have that

$$\begin{aligned}\frac{1}{\zeta_{\mathbb{A}}(2)}\Bigg(& q^{3t+\frac{3}{2}}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!} + q^{3t+1}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-n}q^k + q^{3t+\frac{5}{2}}\sum_{n=0}^{t-1}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-1-n}q^k\\ &+ q^{3t+\frac{7}{2}}\sum_{n=0}^{t-1}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=t-1-n}^{2(t-1-n)}q^k - q^{3t+\frac{5}{2}}\sum_{n=0}^{t-1}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-1-n}q^{2k} + q^{3t+2}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=t-n}^{2(t-n)}q^k\\ &- q^{3t+1}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-n}q^{2k} - q^{3t+\frac{3}{2}}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-n}q^{2k} - q^{3t+2}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-n}q^{2k}\Bigg) = 0.\end{aligned} \tag{A.3}$$

It remains to show that (A.1) is equal to zero for $m = t+1$. For $m = t+1$, we have that (A.1) is equal to

$$\begin{aligned}\frac{1}{\zeta_{\mathbb{A}}(2)}\Bigg(& q^{3t+\frac{9}{2}}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!} + q^{3t+4}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t+1-n}q^k + q^{3t+\frac{11}{2}}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-n}q^k\\ &+ q^{3t+\frac{13}{2}}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=t-n}^{2(t-n)}q^k - q^{3t+\frac{11}{2}}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-n}q^{2k} + q^{3t+5}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=t+1-n}^{2(t+1-n)}q^k\\ &- q^{3t+4}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t+1-n}q^{2k} - q^{3t+\frac{9}{2}}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t+1-n}q^{2k} - q^{3t+5}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t+1-n}q^{2k}\Bigg).\end{aligned} \tag{A.4}$$

Rearranging (A.4), we have that (A.1) is equal to

$$\frac{q^3}{\zeta_{\mathbb{A}}(2)}\Bigg(q^{3t+\frac{3}{2}}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}+q^{3t+1}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-n}q^k+q^{3t+\frac{5}{2}}\sum_{n=0}^{t-1}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-1-n}q^k$$
$$+q^{3t+\frac{7}{2}}\sum_{n=0}^{t-1}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=t-1-n}^{2(t-1-n)}q^k-q^{3t+\frac{5}{2}}\sum_{n=0}^{t-1}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-1-n}q^{2k}+q^{3t+2}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=t-n}^{2(t-n)}q^k$$
$$-q^{3t+1}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-n}q^{2k}-q^{3t+\frac{3}{2}}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-n}q^{2k}-q^{3t+2}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-n}q^{2k}\Bigg) \qquad \text{(A.5)}$$

$$+\frac{1}{\zeta_{\mathbb{A}}(2)}\Bigg(q^{3t+\frac{9}{2}}\frac{\mathcal{C}^{(t+1)}(0)}{(t+1)!}+q^{3t+4}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{t+1-n}+q^{3t+\frac{11}{2}}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{t-n}-q^{3t+\frac{13}{2}}\sum_{n=0}^{t-1}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{t-1-n}$$
$$+q^{3t+\frac{13}{2}}\sum_{n=0}^{t-1}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t-n)-1}(1+q)+q^{3t+\frac{13}{2}}\frac{\mathcal{C}^{(t)}(0)}{t!}-q^{3t+\frac{11}{2}}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t-n)}$$
$$-q^{3t+5}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{t-n}+q^{3t+5}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t-n)+1}(1+q)+q^{3t+5}\frac{\mathcal{C}^{(t+1)}(0)}{(t+1)!}$$
$$-q^{3t+4}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t+1-n)}-q^{3t+\frac{9}{2}}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t+1-n)}-q^{3t+5}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t+1-n)}\Bigg). \qquad \text{(A.6)}$$

Using the inductive hypothesis, we have that (A.5) equals zero, therefore it remains to show that (A.6) equals zero. Rearranging (A.6) we get that it is equal to

$$\frac{1}{\zeta_{\mathbb{A}}(2)}\Bigg(-q^{3t+\frac{13}{2}}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t-n)}+q^{3t+4}\frac{\mathcal{C}^{(t+1)}(0)}{(t+1)!}+q^{3t+\frac{11}{2}}\frac{\mathcal{C}^{(t)}(0)}{t!}+q^{3t+\frac{11}{2}}\sum_{n=0}^{t-1}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t-n)}$$
$$+q^{3t+\frac{13}{2}}\sum_{n=0}^{t-1}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t-n)}+q^{3t+\frac{13}{2}}\frac{\mathcal{C}^{(t)}(0)}{t!}-q^{3t+\frac{11}{2}}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t-n)}+q^{3t+6}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t-n)}$$
$$+q^{3t+7}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t-n)}-q^{3t+7}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t-n)}-q^{3t+4}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t+1-n)}\Bigg)$$
$$=\frac{1}{\zeta_{\mathbb{A}}(2)}\Bigg(q^{3t+\frac{11}{2}}(1+q^{\frac{1}{2}}+q+q^{\frac{3}{2}})\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t-n)}-q^{3t+\frac{11}{2}}(1+q^{\frac{1}{2}}+q+q^{\frac{3}{2}})\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t-n)}\Bigg)=0.$$

Thus (A.1)= 0 for $m=t+1$ and so, by induction, (A.1)= 0 for all $g\geq 1, g$ even.

A.2 g odd

Now let $g=2m+1$, then we want to show, using induction on m that (A.1) equals zero for all $m\geq 0$. For the base case, $m=0$, we have that (A.1) is equal to

$$\frac{1}{\zeta_{\mathbb{A}}(2)}\Bigg(q^{\frac{5}{2}}\big(\mathcal{C}(0)+\mathcal{C}'(0)\big)+q^{\frac{7}{2}}\mathcal{C}(0)+q^3\mathcal{C}(0)+q^4\mathcal{C}(0)-q^3\mathcal{C}(0)+q^{\frac{9}{2}}\mathcal{C}(0)-q^{\frac{7}{2}}\mathcal{C}(0)$$
$$-q^4\mathcal{C}(0)-q^{\frac{5}{2}}\big(\mathcal{C}(0)(1+q^2)+\mathcal{C}'(0)\big)\Bigg),$$

which, when cancelling the terms equals zero, hence the base case is true. Assume that (A.1)= 0 for $m = t$. Then we have that

$$\frac{1}{\zeta_{\mathbb{A}}(2)}\Bigg(q^{3t+\frac{5}{2}}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}+q^{3t+\frac{7}{2}}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-n}q^k+q^{3t+3}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-n}q^k$$
$$+q^{3t+4}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=t-n}^{2(t-n)}q^k-q^{3t+3}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-n}q^{2k}+q^{3t+\frac{9}{2}}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=t-n}^{2(t-n)}q^k$$
$$-q^{3t+\frac{7}{2}}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-n}q^{2k}-q^{3t+4}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-n}q^{2k}-q^{3t+\frac{5}{2}}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t+1-n}q^{2k}\Bigg)=0. \quad (A.7)$$

It remains to show that (A.1) is equal to zero for $m = t + 1$. For $m = t + 1$, we have that (A.1) is equal to

$$\frac{1}{\zeta_{\mathbb{A}}(2)}\Bigg(q^{3t+\frac{11}{2}}\sum_{n=0}^{t+2}\frac{\mathcal{C}^{(n)}(0)}{n!}+q^{3t+\frac{13}{2}}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t+1-n}q^k+q^{3t+6}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t+1-n}q^k$$
$$+q^{3t+7}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=t+1-n}^{2(t+1-n)}q^k-q^{3t+6}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t+1-n}q^{2k}+q^{3t+\frac{15}{2}}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=t+1-n}^{2(t+1-n)}q^k$$
$$-q^{3t+\frac{13}{2}}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t+1-n}q^{2k}-q^{3t+7}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t+1-n}q^{2k}-q^{3t+\frac{11}{2}}\sum_{n=0}^{t+2}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t+2-n}q^{2k}\Bigg). \quad (A.8)$$

Rearranging (A.8), we have that (A.1) is equal to

$$\frac{q^3}{\zeta_{\mathbb{A}}(2)}\Bigg(q^{3t+\frac{5}{2}}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}+q^{3t+\frac{7}{2}}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-n}q^k+q^{3t+3}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-n}q^k$$
$$+q^{3t+4}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=t-n}^{2(t-n)}q^k-q^{3t+3}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-n}q^{2k}+q^{3t+\frac{9}{2}}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=t-n}^{2(t-n)}q^k$$
$$-q^{3t+\frac{7}{2}}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-n}q^{2k}-q^{3t+4}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-n}q^{2k}-q^{3t+\frac{5}{2}}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-n}q^{2k}\Bigg) \quad (A.9)$$

$$+\frac{1}{\zeta_{\mathbb{A}}(2)}\Bigg(q^{3t+\frac{11}{2}}\frac{\mathcal{C}^{(t+2)}(0)}{(t+2)!}+q^{3t+\frac{13}{2}}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{t+1-n}+q^{3t+6}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{t+1-n}-q^{3t+7}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{t-n}$$
$$+q^{3t+7}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t+1-n)-1}(1+q)+q^{3t+7}\frac{\mathcal{C}^{(t+1)}(0)}{(t+1)!}-q^{3t+6}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t+1-n)}$$
$$-q^{3t+\frac{15}{2}}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{t-n}+q^{3t+\frac{15}{2}}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t+1-n)-1}(1+q)+q^{3t+\frac{15}{2}}\frac{\mathcal{C}^{(t+1)}(0)}{(t+1)!}$$
$$-q^{3t+\frac{13}{2}}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t+1-n)}-q^{3t+7}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t+1-n)}-q^{3t+\frac{11}{2}}\sum_{n=0}^{t+2}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t+2-n)}\Bigg). \quad (A.10)$$

Using the inductive hypothesis, we have (A.9) equals zero. Thus it remains to show that (A.10) equals zero. Rearranging (A.10), we have that it equals

$$\frac{1}{\zeta_{\mathbb{A}}(2)}\Bigg(-q^{3t+\frac{15}{2}}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t+1-n)}+q^{3t+\frac{13}{2}}\frac{\mathcal{C}^{(t+1)}(0)}{(t+1)!}+q^{3t+6}\frac{\mathcal{C}^{(t+1)}(0)}{(t+1)!}+q^{3t+6}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t+1-n)}$$
$$+q^{3t+7}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t+1-n)}+q^{3t+7}\frac{\mathcal{C}^{(t+1)}(0)}{(t+1)!}-q^{3t+6}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t+1-n)}$$
$$+q^{3t+\frac{13}{2}}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t+1-n)}+q^{3t+\frac{15}{2}}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t+1-n)}+q^{3t+\frac{15}{2}}\frac{\mathcal{C}^{(t+1)}(0)}{(t+1)!}$$
$$-q^{3t+\frac{13}{2}}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t+1-n)}-q^{3t+7}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t+1-n)}\Bigg)$$
$$=\frac{1}{\zeta_{\mathbb{A}}(2)}\Bigg(q^{3t+6}(1+q^{\frac{1}{2}}+q+q^{\frac{3}{2}})\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t+1-n)}-q^{3t+6}(1+q^{\frac{1}{2}}+q+q^{\frac{3}{2}})\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t+1-n)}\Bigg)=0.$$

Thus (A.1)= 0 for $m=t+1$ and so by induction (A.1)= 0 for all $g\geq 1, g$ odd. This completes the proof of Lemma 5.6.

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