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A NOTE ON THE MEAN VALUE
OF *L*-FUNCTIONS IN FUNCTION FIELDS

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An asymptotic formula for the sum $\sum L(1, \chi)$ is established for a family of hyperelliptic curves of genus g over a fixed finite field $\mathbb{F}_q$ as $g \to \infty$ making use of the analog of the approximate functional equation for such L-functions. As a corollary, we obtain a formula for the average of the class number of the associated rings $\mathbb{F}_q[T, \sqrt{D}]$.

Keywords: Mean values of L-functions; finite fields; function fields; hyperelliptic curves; class numbers.

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1. Introduction

In [5], Gauss presented two conjectures concerning the average values of the class number associated with binary quadratic forms $ax^2 + 2bxy + cy^2$ where $a, b, c \in \mathbb{Z}$. For completeness, clarity and to put our problem in the right context we will restate the Gauss's conjectures.

Let $D = 4(b^2 - ac)$ be the discriminant of the quadratic form $ax^2 + 2bxy + cy$ with $D \equiv 0, 1 \pmod{4}$. Recall that two quadratic forms are equivalent if it is possible to transform the first form into the second through an invertible integral linear change of variables. So we have defined an equivalence relation on the set of quadratic forms and the equivalence classes will be called classes of quadratic forms. Gauss showed that the number of equivalence classes of quadratic forms with discriminant D is finite. Let h_D denote this number, we also call h_D the *class number*. We now present Gauss's conjectures quoted from [6].

Conjecture 1.1 (Gauss). *Let h_D be the class number defined as above. So,*

(1) *Let $D = -4k$ run over all negative discriminants with $k \leq N$. Then*

$$\sum_{1 \leq k \leq N} h_D \sim \frac{4\pi}{21\zeta(3)} N^{\frac{3}{2}}. \tag{1.1}$$

(2) *Let $D = 4k$ run over all positive discriminants such that $k \leq N$. Then*

$$\sum_{1 \leq k \leq N} h_D R_D \sim \frac{4\pi^2}{21\zeta(3)} N^{\frac{3}{2}}, \tag{1.2}$$

where the number R_D is associated to the regulator of the real quadratic number field $\mathbb{Q}(\sqrt{D})$.

The first conjecture (1.1) was proved by Lipschitz [8] and the second (1.2) by Siegel [11]. We now will move to the function field analog of these results and for this let us now define some notation that will be used in the rest of this paper. For more details see [6, 9].

1.1. *Facts about $\mathbb{F}_q[T]$*

The norm of the polynomial $f \in \mathbb{F}_q[T]$ is defined to be $|f| = q^{\deg(f)}$, and we define the zeta function associated to $A = \mathbb{F}_q[T]$ by

$$\zeta_A(s) = \sum_{\substack{f \in A \\ f \text{ monic}}} \frac{1}{|f|^s} \quad (\Re(s) > 1) \tag{1.3}$$

and is easy to show that

$$\zeta_A(s) = \frac{1}{1 - q^{1-s}}. \tag{1.4}$$

We also define the analog of the Möbius function for $A = \mathbb{F}_q[T]$ as follows:

$$\mu(f) = \begin{cases} (-1)^t, & f = \alpha P_1 P_2 \cdots P_t, \\ 0, & \text{otherwise,} \end{cases} \tag{1.5}$$

where each P_j is a distinct monic irreducible polynomial. And the Euler totient function for A, denoted by $\Phi(f)$, is defined to be the number of non-zero polynomials of degree less than $\deg(f)$ and relatively prime to f.

We will assume from now that $q \equiv 1 \pmod 4$ and let $\mathbb{F}_q$ be a finite field with q elements. We denote by $A = \mathbb{F}_q[T]$ the polynomial ring over $\mathbb{F}_q$ and by $k = \mathbb{F}_q(T)$ the rational function field over $\mathbb{F}_q$. Let K/k be a quadratic extension and call $\mathcal{O}_K$ the integral closure of A in K.

Consider that $D \in A$ is a square-free polynomial and put $\mathcal{O}_K$ as $\mathcal{O}_D = A[\sqrt{D}]$. So in this case, $\mathcal{O}_D$ is a Dedekind domain and the associated class number h_D is equal to $|\mathrm{Pic}(\mathcal{O}_D)|$, where $\mathrm{Pic}(\mathcal{O}_D)$ is the Picard group of $\mathcal{O}_D$. For a detailed explanation about the Picard group in this context see [9, p. 315].

Let $D \in A$ be a monic and square-free polynomial. We can define the quadratic character χ_D using the quadratic residue symbol for $\mathbb{F}_q[T]$ by

$$\chi_D(f) = \left(\frac{D}{f}\right), \tag{1.6}$$

and the associated L-function by

$$L(s, \chi_D) = \sum_{\substack{f \in A \\ f \text{ monic}}} \frac{\chi_D(f)}{|f|^s}. \tag{1.7}$$

Now, using [9, Proposition 4.3] we have $L(s, \chi_D)$ is a polynomial in $u = q^{-s}$ of degree at most $\deg(D) - 1$.

Hoffstein and Rosen [5] succeeded in calculating the average value of the class number h_D when the average is taken over all monic polynomials D of a fixed degree, they showed the following.

Theorem 1.2 (Hoffstein and Rosen). *Let M be odd and positive. Then*

$$\frac{1}{q^M} \sum_{\substack{D \text{ monic} \\ \deg(D) = M}} h_D = \frac{\zeta_A(2)}{\zeta_A(3)} q^{\frac{M-1}{2}} - q^{-1}. \tag{1.8}$$

The theorem above can be seen as the function field analog of the Gauss's conjectures. A problem which is more difficult and we consider in this paper is to average the class number over fundamental discriminants, i.e. D monic and square-free. We should note that the calculation presented in this paper follows the same philosophy of the calculations firstly presented by Faifman, Kurlberg and Rudnick in [4, 7, 10], where we fix the number of elements of the finite field and compute the limit as $\deg(D) \to \infty$ to obtain our asymptotic formulas.

2. Statement of Results

The main objective of this paper is to establish an asymptotic formula for

$$\frac{1}{\#\mathcal{H}_{2g+1,q}} \sum_{D \in \mathcal{H}_{2g+1,q}} h_D, \tag{2.1}$$

as $\deg(D) \to \infty$, where h_D is the associated class number and $\mathcal{H}_{2g+1,q}$ denotes the following set:

$$\mathcal{H}_{2g+1,q} = \{D \text{ monic}, \ \deg(D) = 2g + 1, \ D \text{ square-free}, \ D \in A\}. \tag{2.2}$$

The number of elements in $\mathcal{H}_{2g+1,q}$ is given by

$$\#\mathcal{H}_{2g+1,q} = (q - 1)q^{2g}, \tag{2.3}$$

as can be seen using [9, Proposition 2.3].

2.1. *The geometric viewpoint*

As we said earlier, the class number h_D is equal to the $|\text{Pic}(\mathcal{O}_D)|$, where $\text{Pic}(\mathcal{O}_D)$ is the Picard group of $\mathcal{O}_D$. But we should also note that if $D \in \mathcal{H}_{2g+1,q}$, then the equation $y^2 = D(T)$ defines a hyperelliptic curve C_D over $\mathbb{F}_q$ of genus g and the number h_D is closely related to the set of the $\mathbb{F}_q$-rational points on its Jacobian, $\text{Jac}(C_D)$, and so our result also has a geometric appeal. We will now develop this geometric side and then state our results.

The zeta function of the curve C_D over $\mathbb{F}_q$ is a rational function as shown by Weil [12],

$$Z_{C_D}(u) = \frac{P_{C_D}(u)}{(1-u)(1-qu)}, \tag{2.4}$$

where $P_{C_D}(u)$ is a polynomial of degree $2g$ with coefficients in $\mathbb{Z}$. Making use of [9, Proposition 14.6 and Proposition 17.7], we can show that the zeta function of the curve C_D is given by

$$Z_{C_D}(u) = \frac{\mathcal{L}(u, \chi_D)}{(1-u)(1-qu)}, \tag{2.5}$$

where $\mathcal{L}(u, \chi_D) = L(s, \chi_D)$ is the L-function associated with the quadratic character as given in (1.7). We have $\mathcal{L}(u, \chi_D)$ satisfies the following functional equation

$$\mathcal{L}(u, \chi_D) = (qu^2)^g \mathcal{L}\left(\frac{1}{qu}, \chi_D\right), \tag{2.6}$$

as can be seen from [9, Theorem 5.9]. This allows us to state and prove a lemma which will be the starting point of the main calculations of this paper.

Lemma 2.1. *If* $D \in \mathcal{H}_{2g+1,q}$ *we have that* $L(s, \chi_D)$ *can be expressed as follows*:

$$L(s, \chi_D) = \sum_{\substack{f_1 \text{ monic} \\ \deg(f_1) \leq g}} \frac{\chi_D(f_1)}{|f_1|^s} + (q^{1-2s})^g \sum_{\substack{f_2 \text{ monic} \\ \deg(f_2) \leq g-1}} \frac{\chi_D(f_2)}{|f_2|^{1-s}}. \tag{2.7}$$

Proof. Using the ideas presented by Conrey *et al.* [3] we substitute $\mathcal{L}(u, \chi_D) = \sum_{n=0}^{2g} a_n u^n$ into the functional equation (2.6)

$$\sum_{n=0}^{2g} a_n u^n = q^g u^{2g} \sum_{m=0}^{2g} a_m \left(\frac{1}{qu}\right)^m$$

$$= \sum_{m=0}^{2g} a_m q^{g-m} u^{2g-m} = \sum_{k=0}^{2g} a_{2g-k} q^{k-g} u^k.$$

Therefore,

$$\sum_{n=0}^{2g} a_n u^n = \sum_{k=0}^{2g} a_{2g-k} q^{k-g} u^k.$$

Equating coefficients we have that

$$a_n = a_{2g-n}q^{n-g} \quad \text{or} \quad a_{2g-n} = a_n q^{g-n}$$

and so we can write the polynomial $\mathcal{L}(u, \chi_D)$ as

$$\sum_{n=0}^{2g} a_n u^n = \sum_{n=0}^{g} a_n u^n + \sum_{m=0}^{g-1} a_{2g-m} u^{2g-m}$$

$$= \sum_{n=0}^{g} a_n u^n + q^g u^{2g} \sum_{m=0}^{g-1} a_m q^{-m} u^{-m}. \tag{2.8}$$

Writing $a_n = \sum_{\substack{f \text{ monic} \\ \deg(f)=n}} \chi_D(f)$ and $u = q^{-1/2}$ in (2.8) proves the lemma. $\qquad\square$

With this in mind we are ready to state our main results.

Theorem 2.2. *Let $\mathbb{F}_q$ be a fixed finite field with $q \equiv 1$ (mod 4). Then*

$$\sum_{D \in \mathcal{H}_{2g+1,q}} L(1, \chi_D) = |D| \left\{ P(2) - P(1) \left(\frac{1}{q^{[g/2]+1}} + \frac{1}{\zeta_A(2)^2 q^{g[(g-1)/2]}} \right) \right\}$$

$$+ O((2q)^g), \tag{2.9}$$

where $|D| = q^{2g+1}$ and

$$P(s) = \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left(1 - \frac{1}{(|P|+1)|P|^s} \right). \tag{2.10}$$

As a corollary of the Theorem 2.2 we have the following corollary.

Corollary 2.3.

$$\frac{1}{\#\mathcal{H}_{2g+1,q}} \sum_{D \in \mathcal{H}_{2g+1,q}} L(1, \chi_D) \sim \zeta_A(2)P(2) \tag{2.11}$$

as $\deg(D) \to \infty$, i.e. $g \to \infty$.

Proof. Using the Theorem 2.2 together with (2.3) and computing the limit as $g \to \infty$ we can conclude the asymptotic formula above. $\qquad\square$

To obtain a formula for the average of the class number h_D we will make use of the following theorem due to Artin [2].

Theorem 2.4 (Artin). *Let $D \in A$ be a square-free polynomial of degree M. Then if M is odd, we have*

$$L(1, \chi_D) = \frac{\sqrt{q}}{\sqrt{|D|}} h_D. \tag{2.12}$$

We now have all the ingredients to obtain an asymptotic formula for the average of the class number h_D.

Theorem 2.5. *Under the same hypothesis as in Theorem 2.2, we have*

$$\frac{1}{\#\mathcal{H}_{2g+1,q}} \sum_{D \in \mathcal{H}_{2g+1,q}} h_D \sim \frac{\sqrt{|D|}}{\sqrt{q}} \zeta_A(2) P(2), \tag{2.13}$$

as $\deg(D) \to \infty$.

Proof. Straightforward by making use of Corollary 2.3 and Theorem 2.4. $\qquad\square$

3. Preliminary Lemmas

We will require some auxiliary lemmas. Firstly, we will begin by stating a result due to Faifman and Rudnick [4].

Lemma 3.1 (Faifman–Rudnick). *Let χ be a non-trivial Dirichlet character modulo D. Then, for $n < \deg(D)$,*

$$\left| \sum_{\substack{B \text{ monic} \\ \deg(B)=n}} \chi(B) \right| \leq \binom{\deg(D) - 1}{n} q^{n/2}. \tag{3.1}$$

We now prove a bound for non-trivial character sums using Lemma 3.1, which is a consequence of the Riemann Hypothesis for function fields.

Lemma 3.2. *We have that*

(1)

$$\sum_{D \in \mathcal{H}_{2g+1,q}} \sum_{n=0}^{g} q^{-n} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f \neq \square}} \chi_D(f) \ll (2q)^g. \tag{3.2}$$

(2)

$$q^{-g} \sum_{D \in \mathcal{H}_{2g+1,q}} \sum_{m=0}^{g-1} \sum_{\substack{f \text{ monic} \\ \deg(f)=m \\ f \neq \square}} \chi_D(f) \ll (2q)^g. \tag{3.3}$$

Proof. We will establish the part (1) of this lemma. For part (2) we have that the proof is analogous. We start with

$$\sum_{D \in \mathcal{H}_{2g+1,q}} \sum_{n=0}^{g} q^{-n} \sum_{\substack{f \text{ monic} \\ \deg(f)=m \\ f \neq \square}} \chi_D(f)$$

$$= \sum_{n=0}^{g} q^{-n} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f \neq \square}} \sum_{\substack{D \text{ monic} \\ \deg(D)=2g+1}} \sum_{\substack{A \text{ monic} \\ A^2 | D}} \mu(A) \left(\frac{D}{f}\right)$$

$$= \sum_{n=0}^{g} q^{-n} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f \neq \square}} \sum_{\substack{A \text{ monic} \\ \deg(A) \leq g}} \mu(A) \left(\frac{A}{f}\right)^2 \sum_{\substack{B \text{ monic} \\ \deg(B)=2g+1-2\deg(A)}} \left(\frac{B}{f}\right). \quad (3.4)$$

Now we note that (B/f) is a non-trivial character since $f \neq \square$. So we can invoke Lemma 3.1 to get the following bound

$$\left| \sum_{\substack{B \text{ monic} \\ \deg(B)=2g+1-2\deg(A)}} \left(\frac{B}{f}\right) \right| \leq \binom{\deg(f)-1}{2g+1-2\deg(A)} q^{g+\frac{1}{2}-\deg(A)} \quad (3.5)$$

if $2g + 1 - 2\deg(A) < \deg(f)$, and the sum is zero otherwise. So we have

$$\sum_{D \in \mathcal{H}_{2g+1,q}} \sum_{n=0}^{g} q^{-n} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f \neq \square}} \chi_D(f)$$

$$\leq \sum_{n=0}^{g} q^{-n} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f \neq \square}} \sum_{\substack{A \text{ monic} \\ \deg(A) \leq g}} \mu(A) \left| \sum_{\substack{B \text{ monic} \\ \deg(B)=2g+1-2\deg(A)}} \left(\frac{B}{f}\right) \right|$$

$$\leq \sum_{n=0}^{g} q^{-n} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f \neq \square}} \sum_{j=g+\frac{1}{2}-\frac{\deg(f)}{2}}^{g} \sum_{\substack{A \text{ monic} \\ \deg(A)=j}} \binom{\deg(f)-1}{2g+1-2\deg(A)} q^{g+\frac{1}{2}-\deg(A)}$$

$$\ll q^g \sum_{n=0}^{g} q^{-n} \sum_{\substack{f \text{ monic} \\ \deg(f)=n}} \sum_{j=g+\frac{1}{2}-\frac{\deg(f)}{2}}^{g} \binom{\deg(f)-1}{2g+1-2j}$$

$$\ll q^g \sum_{n=0}^{g} q^{-n} \sum_{\substack{f \text{ monic} \\ \deg(f)=n}} 2^{\deg(f)-1} = q^g \sum_{n=0}^{g} 2^n \ll q^g 2^g.$$

$\square$

We now state and prove our next two lemmas.

Lemma 3.3. *For $|D| = q^{2g+1}$ we have*

$$|D| \sum_{\substack{d \text{ monic} \\ \deg(d) > [g/2]}} \frac{\mu(d)}{|d|^2} \prod_{\substack{P \text{ monic} \\ \text{irreducible} \\ P|d}} \frac{1}{|P|+1} \ll q^g. \tag{3.6}$$

Proof.

$$|D| \sum_{\substack{d \text{ monic} \\ \deg(d) > [g/2]}} \frac{\mu(d)}{|d|^2} \prod_{P|d} \frac{1}{|P|+1} \le |D| \sum_{\substack{d \text{ monic} \\ \deg(d) > [g/2]}} \frac{1}{|d|^2} \prod_{P|d} \frac{1}{|P|}$$

$$= |D| \sum_{h > [g/2]} q^{-2h} \ll |D| q^{-g}$$

$$\ll q^g. \qquad \square$$

Lemma 3.4. *We have*

$$\frac{|D|}{\zeta_A(2)} \sum_{\substack{d \text{ monic} \\ \deg(d) \le [g/2]}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} \left(\frac{q^{-\deg(d)}}{1 - q^{-1}} \right)$$

$$= |D| \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left(1 - \frac{1}{(|P|+1)|P|^2} \right) + O(q^g). \tag{3.7}$$

Proof.

$$\frac{|D|}{\zeta_A(2)} \sum_{\substack{d \text{ monic} \\ \deg(d) \le [g/2]}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} \left(\frac{q^{-\deg(d)}}{1 - q^{-1}} \right)$$

$$= |D| \sum_{d \text{ monic}} \frac{\mu(d)}{|d|^2} \prod_{P|d} \frac{1}{|P|+1} - |D| \sum_{\substack{d \text{ monic} \\ \deg(d) > [g/2]}} \frac{\mu(d)}{|d|^2} \prod_{P|d} \frac{1}{|P|+1}. \tag{3.8}$$

Writing the sum over all monic polynomials d as an Euler product and using Lemma 3.3 in the sum over d such that $\deg(d) > [g/2]$ we obtain the desired lemma. $\qquad \square$

Using the same ideas used in the proof of Lemmas 3.3 and 3.4 we can also prove the following lemmas.

Lemma 3.5. *We have*

(1)

$$|D| \sum_{\substack{d \text{ monic} \\ \deg(d) > [g/2]}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} \ll q^g. \tag{3.9}$$

(2)

$$\frac{|D|q^{-g}q^{[(g-1)/2]+1}}{\zeta_A(2)(1-q)} \sum_{\substack{d \text{ monic} \\ \deg(d) > [(g-1)/2]}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} \ll q^g. \qquad (3.10)$$

Lemma 3.6. *We have*

(1)

$$\frac{|D|}{\zeta_A(2)} \sum_{\substack{d \text{ monic} \\ \deg(d) \leq [g/2]}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} \left(\frac{q^{-[g/2]-1}}{1-q^{-1}} \right)$$

$$= |D|q^{-[g/2]-1} \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left(1 - \frac{1}{|P|(|P|+1)} \right) + O(q^g). \qquad (3.11)$$

(2)

$$\frac{|D|q^{-g}}{\zeta_A(2)} \sum_{\substack{d \text{ monic} \\ \deg(d) \leq [(g-1)/2]}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} \left(\frac{q^{[(g-1)/2]+1}}{1-q} \right)$$

$$= \frac{|D|q^{-g}q^{[(g-1)/2]+1}}{\zeta_A(2)(1-q)} \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left(1 - \frac{1}{|P|(|P|+1)} \right) + O(q^g). \qquad (3.12)$$

We present now our last lemma.

Lemma 3.7. *We have*

$$\frac{|D|q^{-g}}{\zeta_A(2)(1-q)} \sum_{\substack{d \text{ monic} \\ \deg(d) \leq [(g-1)/2]}} \mu(d) \prod_{P|d} \frac{1}{|P|+1} \ll gq^g. \qquad (3.13)$$

Proof. We have that

$$\frac{|D|q^{-g}}{\zeta_A(2)(1-q)} \sum_{\substack{d \text{ monic} \\ \deg(d) \leq [(g-1)/2]}} \mu(d) \prod_{P|d} \frac{1}{|P|+1} \ll |D|q^{-g} \sum_{\substack{d \text{ monic} \\ \deg(d) \leq [(g-1)/2]}} \frac{1}{|d|}$$

$$\ll |D|q^{-g}([(g-1)/2]+1)$$

$$\ll gq^g. \qquad \square$$

4. The Main Term

The main result of this section is the following proposition, which will be used to establish the main term of Theorem 2.2.

Proposition 4.1.

$$\sum_{\substack{D \in \mathcal{H}_{2g+1,q} \\ (D,l)=1}} 1 = \frac{|D|}{\zeta_A(2) \prod_{P|l}(1+|P|^{-1})} + O\left(\sqrt{|D|}\frac{\Phi(l)}{|l|}\right). \tag{4.1}$$

We will need the following lemmas.

Lemma 4.2. *Let $V_d = \{D \in \mathbb{F}_q[x] : D \text{ monic}, \deg(D) = d\}$. Then*

$$\#\{D \in V_d : (D,l)=1\} = q^d \frac{\Phi(l)}{|l|}. \tag{4.2}$$

Proof.

$$\#\{D \in V_d : (D,l)=1\} = \sum_{\substack{D \text{ monic} \\ \deg(D)=d \\ (D,l)=1}} 1 = \sum_{\substack{D \text{ monic} \\ \deg(D)=d}} \sum_{h|(D,l)} \mu(h)$$

$$= \sum_{h|l} \mu(h) \sum_{\substack{D \text{ monic} \\ \deg(D)=d \\ h|D}} 1 = \sum_{h|l} \mu(h) \sum_{\substack{m \text{ monic} \\ \deg(m)=d-\deg h}} 1$$

$$= q^d \prod_{\substack{P \\ P|l}} \left(1 - \frac{1}{|P|}\right) = q^d \frac{\Phi(l)}{|l|}, \tag{4.3}$$

where we used [9, Proposition 2.4] in (4.3). □

Lemma 4.3. *We have*

$$\sum_{\substack{Q \text{ monic} \\ \deg(Q) > \frac{2g+1}{2} \\ (Q,l)=1}} \frac{\mu(Q)}{|Q|^2} \ll q^{-1/2} q^{-g}. \tag{4.4}$$

Proof.

$$\sum_{\substack{Q \text{ monic} \\ \deg(Q) > \frac{2g+1}{2} \\ (Q,l)=1}} \frac{\mu(Q)}{|Q|^2} \le \sum_{\substack{Q \text{ monic} \\ \deg(Q) > \frac{2g+1}{2} \\ (Q,l)=1}} \frac{1}{|Q|^2}$$

$$= \sum_{n > \frac{2g+1}{2}} \frac{1}{q^n} \ll q^{-1/2} q^{-g}. \tag{4.5}$$
□

Lemma 4.4. *We have that*

$$\sum_{\substack{Q \text{ monic} \\ \deg(Q) \le \frac{2g+1}{2} \\ (Q,l)=1}} \frac{\mu(Q)}{|Q|^2} = \frac{1}{\zeta_A(2)} \frac{1}{\prod_{P|l}(1-1/|P|^2)} + O(q^{-1/2}q^{-g}). \tag{4.6}$$

Proof.

$$\sum_{\substack{Q \text{ monic} \\ \deg(Q) \leq \frac{2g+1}{2} \\ (Q,l)=1}} \frac{\mu(Q)}{|Q|^2} = \sum_{\substack{Q \text{ monic} \\ (Q,l)=1}} \frac{\mu(Q)}{|Q|^2} - \sum_{\substack{Q \text{ monic} \\ \deg(Q) > \frac{2g+1}{2} \\ (Q,l)=1}} \frac{\mu(Q)}{|Q|^2}$$

$$= \prod_{P \nmid l} \left(1 - \frac{1}{|P|^2}\right) - \sum_{\substack{Q \text{ monic} \\ \deg(Q) > \frac{2g+1}{2} \\ (Q,l)=1}} \frac{\mu(Q)}{|Q|^2}, \qquad (4.7)$$

and

$$\prod_{P \nmid l} \left(1 - \frac{1}{|P|^2}\right) = \prod_{P} \left(1 - \frac{1}{|P|^2}\right) \prod_{P | l} \left(1 - \frac{1}{|P|^2}\right)^{-1}$$

$$= \frac{1}{\zeta_A(2)} \frac{1}{\prod_{P|l}(1 - 1/|P|^2)}. \qquad (4.8)$$

Thus,

$$\sum_{\substack{Q \text{ monic} \\ \deg(Q) \leq \frac{2g+1}{2} \\ (Q,l)=1}} \frac{\mu(Q)}{|Q|^2} = \frac{1}{\zeta_A(2)} \frac{1}{\prod_{P|l}(1 - 1/|P|^2)} - \sum_{\substack{Q \text{ monic} \\ \deg(Q) > \frac{2g+1}{2} \\ (Q,l)=1}} \frac{\mu(Q)}{|Q|^2}, \qquad (4.9)$$

and using the estimate of Lemma 4.3 proves the result. $\qquad \square$

Proof of Proposition 4.1. We have that

$$\sum_{\substack{D \in \mathcal{H}_{2g+1,q} \\ (D,l)=1}} 1 = \sum_{\substack{D \in V_{2g+1} \\ (D,l)=1}} \sum_{Q^2 | D} \mu(Q) = \sum_{\substack{Q \text{ monic} \\ \deg(Q) \leq \frac{2g+1}{2} \\ (Q,l)=1}} \mu(Q) \sum_{\substack{D \in V_{2g+1-2\deg(Q)} \\ (D,l)=1}} 1$$

$$= \sum_{\substack{Q \text{ monic} \\ \deg(Q) \leq \frac{2g+1}{2} \\ (Q,l)=1}} \mu(Q) \#\{D \in V_{2g+1-2\deg(Q)} : (D,l) = 1\}. \qquad (4.10)$$

By Lemma 4.2, we have

$$\sum_{\substack{D \in \mathcal{H}_{2g+1,q} \\ (D,l)=1}} 1 = \sum_{\substack{Q \text{ monic} \\ \deg(Q) \leq \frac{2g+1}{2} \\ (Q,l)=1}} \mu(Q) q^{2g+1-2\deg(Q)} \frac{\Phi(l)}{|l|}$$

$$= |D| \frac{\Phi(l)}{|l|} \sum_{\substack{Q \text{ monic} \\ \deg(Q) \leq \frac{2g+1}{2} \\ (Q,l)=1}} \frac{\mu(Q)}{|Q|^2}. \qquad (4.11)$$

Invoking Lemma 4.4 we obtain

$$
\sum_{\substack{D \in \mathcal{H}_{2g+1,q} \\ (D,l)=1}} 1 = |D| \frac{\Phi(l)}{|l|} \left(\frac{1}{\zeta_A(2)} \frac{1}{\prod_{P|l}(1 - 1/|P|^2)} + O(q^{-1/2}q^{-g}) \right)
$$

$$
= |D| \frac{\Phi(l)}{|l|} \frac{1}{\zeta_A(2)} \frac{1}{\prod_{P|l}(1 - 1/|P|^2)} + O\left(|D| \frac{\Phi(l)}{|l|} q^{-1/2}q^{-g} \right), \quad (4.12)
$$

and using $\frac{\Phi(l)}{|l|} = \prod_{P|l}(1 - |P|^{-1})$, we end up with

$$
\sum_{\substack{D \in \mathcal{H}_{2g+1,q} \\ (D,l)=1}} 1 = \frac{|D|}{\zeta_A(2) \prod_{P|l}(1 + |P|^{-1})} + O\left(\sqrt{|D|} \frac{\Phi(l)}{|l|} \right), \quad (4.13)
$$

which proves Proposition 4.1. □

5. Proof of Theorem 2.2

Our argument in this section follows closely [1]. From (2.7), our main goal is to obtain an asymptotic formula for

$$
\sum_{D \in \mathcal{H}_{2g+1,q}} L(1, \chi_D) = \sum_{D \in \mathcal{H}_{2g+1,q}} \sum_{n=0}^{g} \sum_{\substack{f \text{ monic} \\ \deg(f)=n}} \chi_D(f) q^{-n}
$$

$$
+ q^{-g} \sum_{D \in \mathcal{H}_{2g+1,q}} \sum_{m=0}^{g-1} \sum_{\substack{f \text{ monic} \\ \deg(f)=m}} \chi_D(f). \quad (5.1)
$$

We begin by establishing an asymptotic formula for the first term in the right-hand side of (5.1):

$$
\sum_{D \in \mathcal{H}_{2g+1,q}} \sum_{n=0}^{g} \sum_{\substack{f \text{ monic} \\ \deg(f)=n}} \chi_D(f) q^{-n} = \sum_{n=0}^{g} q^{-n} \sum_{D \in \mathcal{H}_{2g+1,q}} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f=l^2}} \chi_D(f)
$$

$$
+ \sum_{n=0}^{g} q^{-n} \sum_{D \in \mathcal{H}_{2g+1,q}} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f \neq \square}} \chi_D(f). \quad (5.2)
$$

Making use of part (1) of Lemma 3.2 we can write (5.2) as

$$\sum_{D \in \mathcal{H}_{2g+1,q}} \sum_{n=0}^{g} \sum_{\substack{f \text{ monic} \\ \deg(f)=n}} \chi_D(f) q^{-n}$$

$$= \sum_{n=0}^{g} q^{-n} \sum_{D \in \mathcal{H}_{2g+1,q}} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f=l^2}} \chi_D(f) + O((2q)^g). \tag{5.3}$$

For the square terms $f = l^2$ we make use of Proposition 4.1 and we end with

$$\sum_{n=0}^{g} q^{-n} \sum_{D \in \mathcal{H}_{2g+1,q}} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f=l^2}} \chi_D(f)$$

$$= \frac{|D|}{\zeta_A(2)} \sum_{m=0}^{[g/2]} q^{-m} \sum_{\substack{d \text{ monic} \\ \deg(d) \leq m}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} + O(q^{g/2})$$

$$= \frac{|D|}{\zeta_A(2)} \sum_{\substack{d \text{ monic} \\ \deg(d) \leq [g/2]}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} \left(\frac{(q^{-1})^{\deg(d)} - (q^{-1})^{[g/2]+1}}{1-q^{-1}} \right) + O(q^{g/2})$$

$$= \frac{|D|}{\zeta_A(2)} \sum_{\substack{d \text{ monic} \\ \deg(d) \leq [g/2]}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} \left(\frac{q^{-\deg(d)}}{1-q^{-1}} \right)$$

$$- \frac{|D|}{\zeta_A(2)} \sum_{\substack{d \text{ monic} \\ \deg(d) \leq [g/2]}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} \left(\frac{q^{-[g/2]+1}}{1-q^{-1}} \right) + O(q^{g/2}). \tag{5.4}$$

Now for the first term of (5.4) we can use Lemma 3.4 and for the second term we can use Lemma 3.6 and so we end up with the following formula for the square terms:

$$\sum_{n=0}^{g} q^{-n} \sum_{D \in \mathcal{H}_{2g+1,q}} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f=l^2}} \chi_D(f)$$

$$= |D| \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left(1 - \frac{1}{|P|^2(|P|+1)} \right)$$

$$- |D| q^{-[g/2]-1} \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left(1 - \frac{1}{|P|(|P|+1)} \right) + O(q^g). \tag{5.5}$$

Substituting (5.5) in (5.3) we have that

$$
\sum_{D \in \mathcal{H}_{2g+1,q}} \sum_{n=0}^{g} \sum_{\substack{f \text{ monic} \\ \deg(f)=n}} \chi_D(f) q^{-n}
$$

$$
= |D| \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left(1 - \frac{1}{|P|^2(|P|+1)} \right)
$$

$$
- |D| q^{-[g/2]-1} \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left(1 - \frac{1}{|P|(|P|+1)} \right) + O((2q)^g). \qquad (5.6)
$$

For the second term in the right-hand side of (5.1) we mimic the calculations above to end with

$$
q^{-g} \sum_{D \in \mathcal{H}_{2g+1,q}} \sum_{m=0}^{g-1} \sum_{\substack{f \text{ monic} \\ \deg(f)=m}} \chi_D(f)
$$

$$
= \frac{|D| q^{-g}}{\zeta_A(2)} \sum_{\substack{d \text{ monic} \\ \deg(d) \leq [(g-1)/2]}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} \sum_{\deg(d) \leq n \leq [(g-1)/2]} q^n
$$

$$
+ O(q^{[(g-1)/2]}) + O((2q)^g)
$$

$$
= \frac{|D| q^{-g}}{\zeta_A(2)} \sum_{\substack{d \text{ monic} \\ \deg(d) \leq [(g-1)/2]}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} \left(\frac{q^{\deg(d)}}{1-q} \right)
$$

$$
- \frac{|D| q^{-g}}{\zeta_A(2)} \sum_{\substack{d \text{ monic} \\ \deg(d) \leq [(g-1)/2]}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} \left(\frac{q^{[(g-1)/2]+1}}{1-q} \right) + O((2q)^g), \quad (5.7)
$$

where the error $O((2q)^g)$ arises when we consider $f \neq \square$ and using part (2) of Lemma 3.2.

For the first term in (5.7) we use the bound given in Lemma 3.7 and for the second term we have

$$
\frac{|D| q^{-g}}{\zeta_A(2)} \sum_{\substack{d \text{ monic} \\ \deg(d) \leq [(g-1)/2]}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} \left(\frac{q^{[(g-1)/2]+1}}{1-q} \right)
$$

$$
= \frac{|D| q^{-g}}{\zeta_A(2)} \left(\sum_{d \text{ monic}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} \left(\frac{q^{[(g-1)/2]+1}}{1-q} \right) \right)
$$

$$
- \frac{|D| q^{-g}}{\zeta_A(2)} \left(\sum_{\substack{d \text{ monic} \\ \deg(d) > [(g-1)/2]}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} \left(\frac{q^{[(g-1)/2]+1}}{1-q} \right) \right). \qquad (5.8)
$$

And we can use part (2) of Lemma 3.6, and so we have that

$$\frac{|D|q^{-g}}{\zeta_A(2)} \sum_{\substack{d \text{ monic} \\ \deg(d)\le[(g-1)/2]}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} \left(\frac{q^{[(g-1)/2]+1}}{1-q} \right)$$

$$= \frac{|D|q^{-g}q^{[(g-1)/2]+1}}{\zeta_A(2)(1-q)} \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left(1 - \frac{1}{|P|(|P|+1)} \right) + O((2q)^g). \quad (5.9)$$

So we can conclude that

$$q^{-g} \sum_{D\in\mathcal{H}_{2g+1,q}} \sum_{m=0}^{g-1} \sum_{\substack{f \text{ monic} \\ \deg(f)=m}} \chi_D(f)$$

$$= -\frac{|D|q^{-g}q^{[(g-1)/2]+1}}{\zeta_A(2)(1-q)} \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left(1 - \frac{1}{|P|(|P|+1)} \right) + O((2q)^g). \quad (5.10)$$

Putting together Eqs. (5.6) and (5.10) and factoring $|D|$ we have that the proof of Theorem 2.2 is complete. □

Acknowledgments

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