Sieve weight smoothings and moments

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Selberg's sieve

Selberg's sieve upper bound:

$$\sum_{d|(a,m)} \mu(d) \leq \left(\sum_{d|(a,m)} \lambda_d \right)^2,$$

for any $\lambda_d \in \mathbb{R}$ with $\lambda_1 = 1$. Minimize quadratic form

$$\sum_{a \in \mathcal{A}} \left(\sum_{d|(a,m)} \lambda_d \right)^2 = \sum_{\substack{d_1, d_2 | m \\ D = [d_1, d_2]}} \lambda_{d_1} \lambda_{d_2} \cdot \#\{a \in \mathcal{A} : \ D | a\}.$$

Assume λ_d supported on squarefree $d \leq R$, so $D \leq R^2$.

Optimizing (making assumptions on $\mathcal{A}$) yields

$$\lambda_d \approx c \cdot \mu(d) \cdot \left(\frac{\log(R/d)}{\log R} \right)^{\kappa} \quad (d \leq R),$$

Selberg's sieve weight & beyond

$$\lambda_d \approx c \cdot \mu(d) \cdot \left(\frac{\log(R/d)}{\log R}\right)^{\kappa} \quad (d \leq R),$$

κ – *sieve dimension*

Weights λ_d decay smoothly to 0.

The larger the dimension κ, the smoother as $d \to R^-$.

Maynard and Tao: In work on small gaps between primes, used k-dimensional generalization of

$$M_f(n; R) := \sum_{d|n} \mu(d) f\left(\frac{\log d}{\log R}\right),$$

$f : \mathbb{R} \to \mathbb{R}$, support on $(-\infty, 1]$, suff smooth, $f(0) = 1$.

If n has no prime factors $\leq R$ then $M_f(n; R) = 1$.

Why so lucky?

Why does

$$M_f(n;R) := \sum_{d|n} \mu(d) f\left(\frac{\log d}{\log R}\right),$$

come close enough to recognizing primes for certain f, but not when we have a sharp cut-off?
Idea now to study:

$$\sum_{n \le x} M_f(n;R)^k$$

which equals

$$\sum_{\substack{d_1,\ldots,d_k \ge 1 \\ D := [d_1,\ldots,d_k]}} \prod_{j=1}^{k} \mu(d_j) f\left(\frac{\log d_j}{\log R}\right) \cdot \sum_{\substack{n \le x \\ D|n}} 1$$

Why so lucky?

$$M_f(n;R) := \sum_{d|n} \mu(d) f\left(\frac{\log d}{\log R}\right),$$

$$\sum_{n\le x} M_f(n;R)^k = \sum_{\substack{d_1,\dots,d_k\ge 1\\ D:=[d_1,\dots,d_k]}} \prod_{j=1}^{k} \mu(d_j) f\left(\frac{\log d_j}{\log R}\right) \cdot \sum_{\substack{n\le x\\ D|n}} 1$$

$$= x\cdot \mathcal{M}_{f,k}(R) + O((\|f\|_\infty R)^k),$$

where

$$\mathcal{M}_{f,k}(R) := \sum_{d_1,\dots,d_k\ge 1} \frac{\prod_{j=1}^{k} \mu(d_j) f\left(\frac{\log d_j}{\log R}\right)}{[d_1,\dots,d_k]}$$

Why super-smooth f work

Theorem 1. *If $f \in C^A(\mathbb{R})$ with $A > 2 + \frac{1}{2k}\binom{2k}{k}$, and $f, f', \ldots, f^{(A)}$ uniformly bounded then*

$$\frac{1}{x}\sum_{n\le x} M_f(n;R)^{2k} \sim \frac{c_{k,f}}{\log R} \quad \text{as } x/R^k \to \infty.$$

where $c_{k,f} > 0$. Moreover, main contribution from n with all prime factors $> R^\epsilon$:

$$\frac{1}{x}\sum_{\substack{n\le x\\ P^-(n)\le R^\eta}} M_f(n;R)^{2k} \ll_{k,f} \frac{\eta^{2k}}{\log R}.$$

What about the sharp-cutoff sum ?

$$M(n;R) := \sum_{\substack{d|n \\ d\le R}} \mu(d).$$

If $n = p_1p_2\cdots p_k$ with $p_1 < p_2 < \ldots < p_k$ then usually $\log\log p_j \approx j$ for each j. If so, this implies

$$\log(p_1p_2\cdots p_m) \approx e+e^2+\ldots+e^m \approx \frac{e}{e-1}\,e^m \approx \frac{\log p_{m+1}}{1.71828\ldots}.$$

So if $p_1p_2\cdots p_m \le R < p_{m+1}$ then the above sum is

$$M(n;R) := \sum_{d|p_1p_2\cdots p_m} \mu(d) = 0.$$

$$\#\{n \le x : \ M(n;R) \ne 0\} \asymp \frac{x}{(\log R)^{\delta}(\log\log R)^{3/2}},$$

$$\text{where} \ \ \delta := 1 - \frac{1+\log\log 2}{\log 2} = 0.086071332\ldots$$

Proof of proportion of non-zero $M(n;R)$

$$\#\{n \le x : \ M(n;R) \ne 0\} \asymp \frac{x}{(\log R)^{\delta}(\log\log R)^{3/2}}.$$

Proof: If n is even write $n = 2^{\ge 1}m$, and squarefree $d|n$ as $d = 2^{0 \text{ or } 1}D$ with $D|m$ so that

$$\sum_{\substack{d|n \\ d\le R}} \mu(d) = \sum_{\substack{D|m \\ D\le R}} \mu(D) + \sum_{\substack{D|m \\ 2D\le R}} \mu(2D)$$

$$= \sum_{\substack{D|m \\ D\le R}} \mu(D) - \sum_{\substack{D|m \\ D\le R/2}} \mu(D) = \sum_{\substack{D|m \\ R/2<D\le R}} \mu(D).$$

Ford (2008) Proportion of integers with a divisor in $[R/p, R]$ is $\asymp \left(\frac{\log p}{\log R}\right)^{\delta} \cdot \frac{1}{(\log\log R)^{3/2}}$, and a positive proportion of those have exactly one such divisor.

Main *number* theory problem

For $M(n;R) = \sum_{d|n,\ d\le R} \mu(d)$, as $x \to \infty$,

$$\frac{1}{x}\sum_{n\le x} M(n;R)^{2k} \sim_{x\to\infty} \sum_{d_1,\ldots,d_{2k}\le R} \frac{\mu(d_1)\cdots\mu(d_{2k})}{[d_1,\ldots,d_{2k}]}$$
$$\sim c_k(\log R)^{\mathrm{Exponent}(k)}.$$

What is $\mathrm{Exponent}(k)$?

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What is Exponent(k)? BRIAN?

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What is Exponent(k)? BRIAN?

Dress, Iwaniec, Tenenbaum (1983):

$$\mathrm{Exponent}(1) = 0$$

Motohashi (2004): $\mathrm{Exponent}(2) = 2$

de la Bretèche (2001),
Balazard, Naimi, Pétermann (2008):

$$\mathrm{Exponent}(k) \text{ exists for all } k.$$

Function field version

Monic irreducible polynomials in $\mathbb{F}_q[t]$:
Unique factorization, and μ.

$$\mathrm{Poly}_q(n,m;k) := \frac{1}{q^n} \sum_{\substack{N\in\mathbb{F}_q[t]\\ \deg N=n}} \left| \sum_{\substack{M|N\\ \deg M=m}} \mu(M) \right|^{2k}$$

$$= \sum_{\substack{M_1,\ldots,M_{2k}\\ \deg M_j=m}} \mu(M_1)\cdots\mu(M_{2k}) \cdot \frac{1}{q^n} \sum_{\substack{F:\ \deg F=n\\ [M_1,\ldots,M_{2k}]|N}} 1,$$

and when $n \geq 2km$,

$$= \sum_{\substack{M_1,\ldots,M_{2k}\\ \deg M_j=m}} \frac{\mu(M_1)\cdots\mu(M_{2k})}{q^{\deg[M_1,\ldots,M_{2k}]}}.$$

Möbius for permutations

Every $\sigma \in S_N$ is unique product of cycles, $C_1 \dots C_r$.

Divisors are analogous to sub-products of cycles of σ :
These cycles act on subsets $T \subset [N]$, so T is fixed by σ.
If $\sigma\big|_T = C_1 \dots C_\ell$, then $\mu(\sigma\big|_T) = (-1)^\ell$.

Signature $\epsilon(\sigma) = (-1)^{\#\{\text{transpositions to create } \sigma\}}$.
For cycle C, $\epsilon(C) = (-1)^{|C|-1}$, so $\epsilon(\sigma) = (-1)^{N-r}$.
If $|T| = m$ then $\epsilon(\sigma\big|_T) = (-1)^{m-\ell}$, and so

$$\sum_{\substack{T\subset[N],\\ |T|=m,\ \sigma(T)=T}} \mu(\sigma\big|_T) = (-1)^m \sum_{\substack{T\subset[N],\\ |T|=m,\ \sigma(T)=T}} \epsilon(\sigma\big|_T).$$

Eberhart, Ford, Green (2015) implies this sum is non-zero for a proportion $\asymp 1/(\log m)^\delta (\log\log m)^{3/2}$ of the permutations in S_N.

The permutation version

We now wish to study

$$\mathrm{Perm}(N, m; k) := \frac{1}{N!} \sum_{\sigma \in S_N} \left| \sum_{\substack{T \subset [N],\ |T|=m, \\ \sigma(T)=T}} \epsilon(\sigma\big|_T) \right|^{2k}.$$

Try $k = 1$, expand sum to obtain

$$\sum_{\substack{T,U \subset [N], \\ |T|=|U|=m,}} \frac{1}{N!} \sum_{\substack{\sigma \in S_N \\ \sigma(T)=T,\ \sigma(U)=U}} \epsilon(\sigma\big|_T)\epsilon(\sigma\big|_U)$$

Let $R = T \cap U$, then $T = R \cup P$, $U = R \cup Q$, and partition $[N] = P \cup Q \cup R \cup V$. Hence

$$\sigma(V) = V,\ \sigma(Q) = Q,\ \sigma(P) = P,\ \sigma(R) = R$$

Think of σ as an arbitrary permutation on each piece:

Permutation version; second moment

$$R = T \cap U,\ T = R \cup P,\ U = R \cup Q,\ [N] = P \cup Q \cup R \cup V$$

so that

$$\epsilon(\sigma\big|_T) = \epsilon(\rho)\epsilon(\pi) \text{ and } \epsilon(\sigma\big|_U) = \epsilon(\rho)\epsilon(\kappa).$$

$$\mathrm{Perm}(N, m; 1) := \sum_{r=0}^{m} \frac{1}{N!} \sum_{\substack{[N]=P\cup Q\cup R\cup V \\ |R|=r,\ |P|=|Q|=m-r,\ |V|=N+r-2m}}$$

$$\cdot \sum_{\rho\in S_R} \epsilon(\rho)^2 \sum_{\pi\in S_P} \epsilon(\pi) \sum_{\kappa\in S_Q} \epsilon(\kappa) \sum_{\nu\in S_V} 1$$

As $S_N = (1,2)S_N$, we have $\sum_{\xi\in S_N} \epsilon(\xi) = 0$ for $N \geq 2$, so $|P|, |Q| \leq 1$. Hence sum is

$$\sum_{r=m-1}^{m} \frac{1}{N!} \cdot \frac{N!}{r!(m-r)!^2(N+r-2m)!} \cdot r!(N+r-2m)! \ = \ 2.$$

Permutation version; $2k$th moment

If subsets are $T_1, \ldots T_{2k}$, partition into sets R_I, defined recursively (largest first):

$$\bigcap_{i \in I} T_i = \bigcup_{\substack{J \subset [2k] \\ I \subset J}} R_J; \qquad \text{and} \qquad R_\emptyset = [N] - \bigcup_i T_i,$$

So $\{R_I : \ I \subset [2k]\}$ partitions $[N]$, and $\mathrm{Perm}(N, m; k)$ equals

$$\sum_{\substack{r_I \geq 0 \ \forall I \\ \sum_{I:\ i \in I} \ r_I = m}} \sum_{\substack{[N] = \cup_I R_I \\ |R_I| = r_I \ \forall I}} \frac{1}{N!} \prod_{I \subset [2k]} \sum_{\rho_I \in S_{R_I}} \epsilon(\rho_I)^{|I|}.$$

Outer sums $= r_I!$ unless $|I|$ odd, $r_I > 1$, to get 0. This equals $c(m, k)$, the number of sets of $2^{2k} - 1$ non-negative integers $\{r_I : \ \emptyset \neq I \subset [2k]\}$ for which

$$\sum_{I:\ i \in I} r_I = m \text{ for each } i, \text{and } r_I = 0 \text{ or } 1 \text{ if } |I| \text{ is odd.}$$

Permutation version; $2k$th moment

We have if $N \geq 2mk$ then

$$\mathrm{Perm}(N, m; k) = c(m, k),$$

the $\#\{(r_I : \emptyset \neq I \subset [2k])\} \subset \mathbb{N}^{2^{2k}-1}$ for which

$$\sum_{I:\ i \in I} r_I = m \text{ for each } i, \text{and } r_I = 0 \text{ or } 1 \text{ if } |I| \text{ is odd.}$$

One has

$$c(m, k) \asymp m^{2^{2k-1}-2k-1} + 1.$$

Also coeff of $(x_1 x_2 \ldots x_{2k})^m$ in

$$\frac{\prod_{|I| \text{ odd}} (1 - \prod_{i \in I} x_i)}{\prod_{|I| \text{ even, } >0} (1 - \prod_{i \in I} x_i)} = \prod_{I \neq \emptyset:\ I \subset [2k]} \left(1 - \prod_{i \in I} x_i\right)^{(-1)^{|I|-1}}$$

("-" not "+" in numerator terms ok for parity reasons)

Back to Function field version

If $n \geq 2km$

$$\mathrm{Poly}_q(n,m;k) = \sum_{\substack{M_1,\ldots,M_{2k} \\ \deg M_j = m}} \frac{\mu(M_1)\cdots\mu(M_{2k})}{q^{\deg[M_1,\ldots,M_{2k}]}} \ .$$

Lcm a pain. Break up M_i's into gcds: Given M_i's define M_I for non-empty $I \subset [2k]$, by induction on $2k - |I|$:

$$M_{[2k]} = \gcd(M_1, \ldots, M_{2k}).$$

$$\text{Otherwise} \quad M_I = \gcd(M_i : \ i \in I) / \prod_{I \subsetneq J \subset [2k]} M_J.$$

Then M_I are pairwise coprime (as M_i are squarefree) and each

$$M_i = \prod_{I:\ i \in I} M_I$$

Function field version, 2

We have M_I are pairwise coprime (as M_i are squarefree) and each $M_i = \prod_{I:\ i \in I} M_I$, so

$$\mathrm{Poly}_q(n, m; k) = \sum_{\substack{M_1, \ldots, M_{2k} \\ \deg M_j = m}} \frac{\mu(M_1) \cdots \mu(M_{2k})}{q^{\deg[M_1, \ldots, M_{2k}]}}$$

$$= \sum_{\substack{M_I \\ \sum_{j \in I} \deg M_I = m \ \forall j}} \prod_{I \subset [2k]} \frac{\mu(M_I)^{|I|}}{q^{\deg M_I}};$$

which is the coefficient of $(x_1 x_2 \ldots x_{2k})^m$ in

$$\sum_{M_I} \prod_{I \subset [2k]} \mu(M_I)^{|I|} \left(\frac{\prod_{i \in I} x_i}{q} \right)^{\deg M_I}$$

Function field version, 3

FTA then gives

$$\sum_{M_I} \prod_{I \subset [2k]} \mu(M_I)^{|I|} \left(\frac{\prod_{i \in I} x_i}{q} \right)^{\deg M_I} =$$

$$\prod_{P \text{ irreducible}} \left(1 + \frac{1}{q^{\deg P}} \left(\sum_{I \neq \emptyset:\ I \subset [2k]} (-1)^{|I|} \left(\prod_{i \in I} x_i \right)^{\deg P} \right) \right).$$

Main term is

$$\prod_{I \neq \emptyset:\ I \subset [2k]} \prod_{P \text{ irreducible}} \left(1 - \left(\frac{\prod_{i \in I} x_i}{q} \right)^{\deg P} \right)^{(-1)^{|I|-1}}.$$

Function field version, 4

Now using the function field zeta function identity

$$\prod_{P \text{ irreducible}} \left(1 - x^{\deg P}\right)^{-1} = (1 - qx)^{-1},$$

we are looking for the coefficient of $(x_1 x_2 \dots x_{2k})^m$ in

$$\prod_{I \neq \emptyset:\ I \subset [2k]} \prod_{P \text{ irreducible}} \left(1 - \left(\frac{\prod_{i \in I} x_i}{q}\right)^{\deg P}\right)^{(-1)^{|I|-1}}$$

$$= \prod_{I \neq \emptyset:\ I \subset [2k]} \left(1 - \prod_{i \in I} x_i\right)^{(-1)^{|I|-1}},$$

which is how $c(m, k)$ was defined!

We have therefore proved that if $n \geq 2mk$ then

$$\mathrm{Poly}_q(n, m; k) = c(m, k)(1 + O_k(1/q)).$$

Theorems

We have if $N \geq 2mk$ then

$$\mathrm{Perm}(N, m; k) = c(m, k),$$

where

$$c(m, k) \asymp m^{2^{2k-1}-2k-1} + 1.$$

If $n \geq 2mk$ then

$$\mathrm{Poly}_q(n, m; k) = c(m, k)(1 + O_k(1/q)).$$

What is the exponent for the question in the integers?

Approaching the number theory problem

For $M(n;R) = \sum_{d|n,\ d\le R} \mu(d)$, as $x/R^{2k} \to \infty$,

$$\frac{1}{x}\sum_{n\le x} M(n;R)^{2k} \sim \sum_{d_1,\ldots,d_k\le R} \frac{\mu(d_1)\cdots\mu(d_{2k})}{[d_1,\ldots,d_{2k}]}.$$

Lcm a pain. Break up d_i's into gcds: Given d_i's define d_I for non-empty $I \subset [2k]$, by induction on $2k - |I|$:

$$d_{[2k]} = \gcd(d_1,\ldots,d_{2k}).$$

$$\text{Otherwise} \quad d_I = \gcd(d_i :\ i \in I) \Big/ \prod_{I \subsetneq J \subset [2k]} d_J.$$

Then d_I are pairwise coprime (as d_i are squarefree) and each

$$d_i = \prod_{I:\ i\in I} d_I$$

Approaching our problem, 2

$$\frac{1}{x}\sum_{n\le x} M(n;R)^{2k} \sim \sum_{d_1,\dots,d_k\le R} \frac{\mu(d_1)\cdots\mu(d_{2k})}{[d_1,\dots,d_{2k}]}$$
$$= \sideset{}{'}\sum_{d_I\ge 1} \prod_{I\subset[2k]} \frac{\mu(d_I)^{|I|}}{d_I}$$

where d_I are pairwise coprime and $d_i = \prod_{I:\ i\in I} d_I \le R$. Use Perron to get (changing d_I to m_I)

$$\frac{1}{(2i\pi)^{2k}} \int\cdots\int_{\substack{\Re(s_j)=\lambda_j/\log R\\ 1\le j\le 2k}} \prod_{I\subset[2k]} \sum_{m_I\ge 1} \frac{\mu(m_I)^{|I|}}{m_I^{1+s_I}} \prod_j \frac{R^{s_j}}{s_j} ds_j$$

where $s_I := \sum_{i\in I} s_i$. We see $\zeta(1+s_I)^{\pm 1}$ factors:

"+" if $|I|$ is even, "-" if $|I|$ is odd.

Being precise about the zeta-function

$$\prod_{\emptyset \neq I \subset [2k]} \sum_{\substack{m_I \geq 1 \\ m_1 \text{ pairwise coprime}}} \frac{\mu(m_I)^{|I|}}{m_I^{1+s_I}}$$

$$= \prod_{p \text{ prime}} \left(1 + \sum_{\emptyset \neq I \subset [2k]} \frac{(-1)^{|I|}}{p^{1+s_I}} \right).$$

The factor at each prime p therefore looks like

$$= \prod_{\emptyset \neq I \subset [2k]} \left(1 - \frac{1}{p^{1+s_I}} \right)^{-(-1)^{|I|}}$$

plus terms $1/p^{2+\cdots}$. Hence our zeta-function equals

$$\frac{\prod_{|I| \text{ even}} \zeta(1+s_I)}{\prod_{|I| \text{ odd}} \zeta(1+s_I)} \cdot E(s_1, \ldots, s_{2k})$$

where $E(s_1, \ldots, s_{2k})$ is abs cvgent if each $\mathrm{Re}(s_i) > \epsilon$.

Approaching our problem, 3

Ignore gcd of M_I's (2ndary terms); integral becomes

$$\frac{1}{(2i\pi)^{2k}} \int\cdots\int_{\substack{\Re(s_j)=\lambda_j/\log R \\ 1\le j\le 2k}} \frac{\prod_{|I|\text{ even}}\zeta(1+s_I)}{\prod_{|I|\text{ odd}}\zeta(1+s_I)} R^{s_{[2k]}} \prod_j \frac{ds_j}{s_j}$$

where the products are over non-empty subsets I of $[2k]$. The plan is to move contours to the left and pick up the ($2k$-dimensional) poles. If some $s_j = 0$ at the pole then: $\zeta(1+s_I)/\zeta(1+s_{I\cup\{j\}}) \to 1$ whenever $j \notin I$, and $s_j\zeta(1+s_j) \to 1$, as $s_j \to 0$, so no contribution.
The integrand is now $\prod_{i\neq j} R^{s_i}/s_i$, and so the pole must be at $(0,0,\ldots,0)$, a multi-pole of order $2k-1$.

$$(\text{Power of } \log R) = (\text{Order of Pole}) - 2k = -1.$$

So this contributes a $c/\log R$ term. Where else?

Approaching our problem, 4

Need to find highest order pole of

$$\frac{\prod_{|I| \text{ even}} \zeta(1+s_I)}{\prod_{|I| \text{ odd}} \zeta(1+s_I)} \cdot \frac{R^{s_{[2k]}}}{s_1 \cdots s_{2k}}.$$

with all $s_i \neq 0$. Order of pole equals

$$\#\{I \subset [2k],\, |I| \text{ even}:\ s_I = 0\} - \#\{I \subset [2k]\, |I| \text{ odd}:\ s_I = 0\} - 1.$$

Maximum is achieved if and only if $s_i = s (\neq 0)$ for k values of i, and $s_j = -s$ for the other k values of j. Then the power of $\log R$ is

$$\binom{2k}{k} - 0 - 1 - (2k-1) = \binom{2k}{k} - 2k$$

(the poles come from shitfing all but s_{2k}).

Number theoretic results

One can use this to prove

$$\frac{1}{x}\sum_{n\le x} M(n;R)^{2k} \sim c_k(\log R)^{\binom{2k}{k}-2k}$$

where c_k is some non-zero constant. That is,

$$\text{Exponent}(k) = \binom{2k}{k} - 2k.$$

More generally:

Theorem 2. *Let*

$$f_A(t) := \begin{cases} (1-t)^A & \textit{for } t \le 1; \\ 0 & \textit{otherwise}, \end{cases}$$

which is A-times differentiable on $\mathbb{R}$. Then

$$\frac{1}{x}\sum_{n\le x} M_{f_A}(n;R)^{2k} \sim c_{k,A}(\log R)^{\binom{2k}{k}-2k(A+1)} + \frac{c_{k,A}}{\log R},$$

where $c_{k,A}$ is some non-zero constant.

Comparing results for Möbius moments

For the permutation moments, and the polynomials in finite field moments, the exponent is

$$2^{2k-1} - 2k - 1.$$

For the integer moments, the exponent is

$$\binom{2k}{k} - 2k.$$

???

Comparing results for Möbius moments

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???

Much seems to depend on whether ζ-poles in the denominator of the zeta-function cancel those in the numerator. For quadratic $\chi \pmod q$, try

$$\frac{1}{x} \sum_{n \le x} \left| \sum_{d|n,\ d \le R} \chi(d) \right|^{2k}$$

Quadratic character $\chi \pmod q$, moments

$$\frac{1}{x}\sum_{n\le x}\left|\sum_{d|n,\ d\le R}\chi(d)\right|^{2k}$$

Integrand product – look for poles in

$$\prod_{|I|\text{ even},>0} L(1+s_I,\chi_0)\cdot\prod_{|I|\text{ odd}} L(1+s_I,\chi).$$

Pole at $(0,\ldots,0)$ order $2^{2k-1}-1$.

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Pole at $(0,\ldots,0)$ order $2^{2k-1}-1$.

If χ has a Siegel zero then $L(1+s_I,\chi)\approx\zeta(1+s_I)^{-1}$ in a limited range. Asymptotic:

$$c_1\,L(1,\chi)^{2^{2k-1}}\cdot\left(\frac{\phi(q)}{q}\log R\right)^{2^{2k-1}-1}+c_2(q)(\log R)^{\binom{2k}{k}-2k}.$$