Sieve weight smoothings and moments

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Selberg’s sieve upper bound:

$$\sum_{d \mid (a,m)} \mu(d) \leq \left( \sum_{d \mid (a,m)} \lambda_d \right)^2,$$

for any \( \lambda_d \in \mathbb{R} \) with \( \lambda_1 = 1 \). Minimize quadratic form

$$\sum_{a \in \mathcal{A}} \left( \sum_{d \mid (a,m)} \lambda_d \right)^2 = \sum_{d_1, d_2 \mid m} \lambda_{d_1} \lambda_{d_2} \cdot \#\{a \in \mathcal{A} : D \mid a\}.$$

Assume \( \lambda_d \) supported on squarefree \( d \leq R \), so \( D \leq R^2 \).

Optimizing (making assumptions on \( \mathcal{A} \)) yields

$$\lambda_d \approx c \cdot \mu(d) \cdot \left( \frac{\log(R/d)}{\log R} \right)^\kappa \quad (d \leq R),$$
Selberg’s sieve weight & beyond

\[ \lambda_d \approx c \cdot \mu(d) \cdot \left( \frac{\log(R/d)}{\log R} \right)^\kappa \quad (d \leq R), \]

\(\kappa - \text{sieve dimension}\)
Weights \(\lambda_d\) decay smoothly to 0.
The larger the dimension \(\kappa\), the smoother as \(d \to R^−\).

**Maynard and Tao:** In work on small gaps between primes, used \(k\)-dimensional generalization of

\[ M_f(n; R) := \sum_{d|n} \mu(d) f \left( \frac{\log d}{\log R} \right), \]

\(f : \mathbb{R} \to \mathbb{R}, \text{ support on } (-\infty, 1], \text{ suff smooth, } f(0) = 1.\)

If \(n\) has no prime factors \(\leq R\) then \(M_f(n; R) = 1.\)
Why so lucky?

Why does

$$M_f(n; R) := \sum_{d|n} \mu(d) f \left( \frac{\log d}{\log R} \right),$$

come close enough to recognizing primes for certain $f$, but not when we have a sharp cut-off?

Idea now to study:

$$\sum_{n \leq x} M_f(n; R)^k$$

which equals

$$\sum_{d_1, \ldots, d_k \geq 1 \atop D:=[d_1, \ldots, d_k]} \prod_{j=1}^k \mu(d_j) f \left( \frac{\log d_j}{\log R} \right) \cdot \sum_{n \leq x \atop D|n} 1$$
Why so lucky?

\[ M_f(n; R) := \sum_{d|n} \mu(d) f \left( \frac{\log d}{\log R} \right), \]

\[ \sum_{n \leq x} M_f(n; R)^k = \sum_{\substack{d_1, \ldots, d_k \geq 1 \\ D := [d_1, \ldots, d_k]}} \prod_{j=1}^{k} \mu(d_j) f \left( \frac{\log d_j}{\log R} \right) \cdot \sum_{n \leq x} \frac{1}{D|n} 
\]

\[ = x \cdot M_{f,k}(R) + O(\| f \|_\infty R^k), \]

where

\[ M_{f,k}(R) := \sum_{d_1, \ldots, d_k \geq 1} \prod_{j=1}^{k} \mu(d_j) f \left( \frac{\log d_j}{\log R} \right) \]
Sieve weights

**Why super-smooth $f$ work**

**Theorem 1.** If $f \in C^A(\mathbb{R})$ with $A > 2 + \frac{1}{2k}(\binom{2k}{k})$, and $f, f', \ldots, f^{(A)}$ uniformly bounded then

$$
\frac{1}{x} \sum_{n \leq x} M_f(n; R)^{2k} \sim \frac{c_{k,f}}{\log R} \quad \text{as} \quad x/R^k \to \infty.
$$

where $c_{k,f} > 0$. Moreover, main contribution from $n$ with all prime factors $> R^\varepsilon$:

$$
\frac{1}{x} \sum_{\substack{n \leq x \\ P^{-}(n) \leq R^n}} M_f(n; R)^{2k} \ll_{k,f} \frac{\eta^{2k}}{\log R}.
$$
What about the sharp-cutoff sum?

\[ M(n; R) := \sum_{d \mid n, \quad d \leq R} \mu(d). \]

If \( n = p_1 p_2 \cdots p_k \) with \( p_1 < p_2 < \ldots < p_k \) then usually \( \log \log p_j \approx j \) for each \( j \). If so, this implies

\[ \log(p_1 p_2 \cdots p_m) \approx e + e^2 + \ldots + e^m \approx \frac{e}{e - 1} e^m \approx \frac{\log p_{m+1}}{1.71828\ldots}. \]

So if \( p_1 p_2 \cdots p_m \leq R < p_{m+1} \) then the above sum is

\[ M(n; R) := \sum_{d \mid p_1 p_2 \cdots p_m} \mu(d) = 0. \]

\[ \#\{n \leq x : M(n; R) \neq 0\} \approx \frac{x}{(\log R)^{\delta}(\log \log R)^{3/2}}, \]

where \( \delta := 1 - \frac{1 + \log \log 2}{\log 2} = 0.086071332\ldots \)
Proof of proportion of non-zero $M(n; R)$

$$\#\{n \leq x : M(n; R) \neq 0\} \asymp \frac{x}{(\log R)^\delta (\log \log R)^{3/2}}.$$  

**Proof:** If $n$ is even write $n = 2^{\geq 1}m$, and squarefree $d|n$ as $d = 2^0$ or $1D$ with $D|m$ so that

$$\sum_{d|n \leq R} \mu(d) = \sum_{D|m \leq R} \mu(D) + \sum_{D|m \leq R} \mu(2D)$$

$$= \sum_{D|m \leq R} \mu(D) - \sum_{D|m \leq R/2} \mu(D) = \sum_{D|m \leq R/2} \mu(D).$$

**Ford (2008)** Proportion of integers with a divisor in $[R/p, R]$ is $\asymp \left(\frac{\log p}{\log R}\right)^\delta \cdot \frac{1}{(\log \log R)^{3/2}}$, and a positive proportion of those have exactly one such divisor.
Main number theory problem

For \( M(n; R) = \sum_{d \mid n, \, d \leq R} \mu(d) \), as \( x \to \infty \),

\[
\frac{1}{x} \sum_{n \leq x} M(n; R)^{2k} \sim_{x \to \infty} \sum_{d_1, \ldots, d_{2k} \leq R} \frac{\mu(d_1) \cdots \mu(d_{2k})}{[d_1, \ldots, d_{2k}]} \sim c_k (\log R)^{\text{Exponent}(k)}.
\]

What is \( \text{Exponent}(k) \)?
Main number theory problem

For \( M(n; R) = \sum_{d|n, \ d \leq R} \mu(d) \), as \( x \to \infty \),

\[
\frac{1}{x} \sum_{n \leq x} M(n; R)^{2k} \sim_{x \to \infty} \sum_{d_1, \ldots, d_{2k} \leq R} \frac{\mu(d_1) \cdots \mu(d_{2k})}{[d_1, \ldots, d_{2k}]} \sim c_k (\log R)^{\text{Exponent}(k)}.
\]

What is \( \text{Exponent}(k) \)? BRIAN?
Main number theory problem

For \( M(n; R) = \sum_{d|n, d \leq R} \mu(d) \), as \( x \rightarrow \infty \),

\[
\frac{1}{x} \sum_{n \leq x} M(n; R)^{2k} \sim_{x \rightarrow \infty} \sum_{d_1, \ldots, d_{2k} \leq R} \frac{\mu(d_1) \cdots \mu(d_{2k})}{[d_1, \ldots, d_{2k}]} \sim c_k (\log R)^{\text{Exponent}(k)}.
\]

What is \( \text{Exponent}(k) \)?

Dress, Iwaniec, Tenenbaum (1983):
\( \text{Exponent}(1) = 0 \)

Motohashi (2004): \( \text{Exponent}(2) = 2 \)

de la Bretèche (2001), Balazard, Naimi, Pétermann (2008):
\( \text{Exponent}(k) \) exists for all \( k \).
Function field version

Monic irreducible polynomials in $\mathbb{F}_q[t]$: Unique factorization, and $\mu$.

$$
\text{Poly}_q(n, m; k) := \frac{1}{q^n} \sum_{N \in \mathbb{F}_q[t]} \sum_{\substack{\mu(M) \mid M|N \\text{deg } M=m \text{deg } N=n}} \mu(M)
$$

$$
= \sum_{M_1, \ldots, M_{2k} \\text{deg } M_j=m} \mu(M_1) \cdots \mu(M_{2k}) \cdot \frac{1}{q^n} \sum_{F: \deg F=n} 1, \quad [M_1, \ldots, M_{2k}] \mid N
$$

and when $n \geq 2km$,

$$
= \sum_{M_1, \ldots, M_{2k} \\text{deg } M_j=m} \frac{\mu(M_1) \cdots \mu(M_{2k})}{q^{\deg [M_1, \ldots, M_{2k}]}}.
$$
Möbius for permutations

Every $\sigma \in S_N$ is unique product of cycles, $C_1 \ldots C_r$.

*Divisors* are analogous to sub-products of cycles of $\sigma$:
These cycles act on subsets $T \subset [N]$, so $T$ is fixed by $\sigma$.
If $\sigma|_T = C_1 \ldots C_\ell$, then $\mu(\sigma|_T) = (-1)^\ell$.

*Signature* $\epsilon(\sigma) = (-1)^\#\{\text{transpositions to create } \sigma\}$.
For cycle $C$, $\epsilon(C) = (-1)^{|C|-1}$, so $\epsilon(\sigma) = (-1)^{N-r}$.
If $|T| = m$ then $\epsilon(\sigma|_T) = (-1)^{m-\ell}$, and so
$$\sum_{T \subset [N], \ |T| = m, \ \sigma(T) = T} \mu(\sigma|_T) = (-1)^m \sum_{T \subset [N], \ |T| = m, \ \sigma(T) = T} \epsilon(\sigma|_T).$$

Eberhart, Ford, Green (2015) implies this sum
is non-zero for a proportion $\asymp 1/(\log m)^\delta (\log \log m)^{3/2}$
of the permutations in $S_N$. 
The permutation version

We now wish to study

\[ \text{Perm}(N, m; k) := \frac{1}{N!} \sum_{\sigma \in S_N} \sum_{T \subset [N], |T| = m, \sigma(T) = T} \epsilon(\sigma|_T) \epsilon(\sigma|_U) \cdot 2^k \]

Try \( k = 1 \), expand sum to obtain

\[ \sum_{T, U \subset [N], |T| = |U| = m, \sigma(T) = T, \sigma(U) = U} \frac{1}{N!} \sum_{\sigma \in S_N} \epsilon(\sigma|_T) \epsilon(\sigma|_U) \]

Let \( R = T \cap U \), then \( T = R \cup P, U = R \cup Q \), and partition \( [N] = P \cup Q \cup R \cup V \). Hence

\[ \sigma(V) = V, \sigma(Q) = Q, \sigma(P) = P, \sigma(R) = R \]

Think of \( \sigma \) as an arbitrary permutation on each piece:
Permutation version; second moment

\[ R = T \cap U, \ T = R \cup P, \ U = R \cup Q, \ [N] = P \cup Q \cup R \cup V \]
so that

\[ \epsilon(\sigma|_T) = \epsilon(\rho)\epsilon(\pi) \text{ and } \epsilon(\sigma|_U) = \epsilon(\rho)\epsilon(\kappa). \]

\[ \text{Perm}(N, m; 1) := \sum_{r=0}^{m} \frac{1}{N!} \sum_{[N]=P\cup Q\cup R\cup V} \text{ } |R|=r, \ |P|=|Q|=m-r, \ |V|=N+r-2m \]

\[ \cdot \sum_{\rho \in S_R} \epsilon(\rho)^2 \sum_{\pi \in S_P} \epsilon(\pi) \sum_{\kappa \in S_Q} \epsilon(\kappa) \sum_{\nu \in S_V} 1 \]

As \( S_N = (1, 2)S_N \), we have \( \sum_{\xi \in S_N} \epsilon(\xi) = 0 \) for \( N \geq 2 \), so \( |P|, |Q| \leq 1 \). Hence sum is

\[ \sum_{r=m-1}^{m} \frac{1}{N!} \frac{N!}{r!(m-r)!(N+r-2m)!} \cdot r!(N+r-2m)! = 2. \]
**Sieve weights**

**Permutation version; 2kth moment**

If subsets are $T_1, \ldots T_{2k}$, partition into sets $R_I$, defined recursively (largest first):

$$\bigcap_{i \in I} T_i = \bigcup_{J \subseteq [2k]} R_J; \quad \text{and} \quad R_\emptyset = [N] - \bigcup_i T_i,$$

So $\{R_I : I \subseteq [2k]\}$ partitions $[N]$, and $\text{Perm}(N, m; k)$ equals

$$\sum_{\sum_I: i \in I \ r_I=m} \sum_{r_I \geq 0 \ \forall I} \frac{1}{N!} \prod_{I \subseteq [2k]} \sum_{\rho_I \in S_{R_I}} \epsilon(\rho_I)^{|I|}.$$

Outer sums $= r_I!$ unless $|I|$ odd, $r_I > 1$, to get 0. This equals $c(m, k)$, the number of sets of $2^{2k} - 1$ non-negative integers $\{r_I : \emptyset \neq I \subseteq [2k]\}$ for which

$$\sum_{I: i \in I} r_I = m \text{ for each } i, \text{ and } r_I = 0 \text{ or } 1 \text{ if } |I| \text{ is odd}.$$. 
Permutation version; 2kth moment

We have if \( N \geq 2mk \) then

\[
\text{Perm}(N, m; k) = c(m, k),
\]

the \(#\{(r_I : \emptyset \neq I \subset [2k])\} \subset \mathbb{N}^{2^{2k}-1}\) for which

\[
\sum_{I: i \in I} r_I = m \text{ for each } i, \text{ and } r_I = 0 \text{ or } 1 \text{ if } |I| \text{ is odd}.
\]

One has

\[
c(m, k) \asymp m^{2^{2k-1}-2k-1+1}.
\]

Also coeff of \((x_1x_2\ldots x_{2k})^m\) in

\[
\frac{\prod_{|I| \text{ odd}} (1 - \prod_{i \in I} x_i)}{\prod_{|I| \text{ even}, >0} (1 - \prod_{i \in I} x_i)} = \prod_{I \neq \emptyset: I \subset [2k]} \left(1 - \prod_{i \in I} x_i\right)^{(-1)^{|I|-1}}
\]

("-" not "+") in numerator terms ok for parity reasons)
Back to Function field version

If \( n \geq 2km \)

\[
\text{Poly}_q(n, m; k) = \sum_{\substack{M_1, \ldots, M_{2k} \text{ deg } M_j = m \text{ deg } \deg[M_1, \ldots, M_{2k}]}} \frac{\mu(M_1) \cdots \mu(M_{2k})}{q^{\deg[M_1, \ldots, M_{2k}]}}.
\]

Lcm a pain. Break up \( M_i \)'s into gcgs: Given \( M_i \)'s define \( M_I \) for non-empty \( I \subset [2k] \), by induction on \( 2k - |I| \):

\[
M_{2k} = \gcd(M_1, \ldots, M_{2k}).
\]

Otherwise \( M_I = \gcd(M_i : i \in I) / \prod_{I \subsetneq J \subset [2k]} M_J. \)

Then \( M_I \) are pairwise coprime (as \( M_i \) are squarefree) and each

\[
M_i = \prod_{I : i \in I} M_I
\]
Function field version, 2

We have $M_I$ are pairwise coprime (as $M_i$ are squarefree) and each $M_i = \prod_{i \in I} M_I$, so

$$\text{Poly}_q(n, m; k) = \sum_{M_1,\ldots,M_{2k} \atop \deg M_j = m} \frac{\mu(M_1) \cdots \mu(M_{2k})}{q^{\deg[M_1,\ldots,M_{2k}]}}$$

$$= \sum_{M_I \atop \sum_{j \in I} \deg M_I = m \; \forall j} \prod_{I \subset [2k]} \frac{\mu(M_I)^{|I|}}{q^{\deg M_I}};$$

which is the coefficient of $(x_1 x_2 \ldots x_{2k})^m$ in

$$\sum \prod_{M_I \atop I \subset [2k]} \mu(M_I)^{|I|} \left(\frac{\prod_{i \in I} x_i}{q}\right)^{\deg M_I}$$
Sieve weights

**Function field version, 3**

FTA then gives

$$\sum_{M_I \ I \subset [2k]} \prod \mu(M_I)^{|I|} \left( \frac{\prod_{i \in I} x_i}{q} \right)^{\deg M_I} =$$

$$\prod_{P \text{ irreducible}} \left( 1 + \frac{1}{q^{\deg P}} \right) \left( \sum_{I \neq \emptyset: I \subset [2k]} (-1)^{|I|} \left( \prod_{i \in I} x_i \right)^{\deg P} \right) .$$

Main term is

$$\prod_{I \neq \emptyset: I \subset [2k]} \prod_{P \text{ irreducible}} \left( 1 - \left( \frac{\prod_{i \in I} x_i}{q} \right)^{\deg P} \right)^{(-1)^{|I|-1}}.$$
Function field version, 4

Now using the function field zeta function identity

$$\prod_{P \text{ irreducible}} (1 - x^{\deg P})^{-1} = (1 - qx)^{-1},$$

we are looking for the coefficient of \((x_1x_2\ldots x_{2k})^m\) in

$$\prod_{I \neq \emptyset: I \subset [2k]} \prod_{P \text{ irreducible}} \left(1 - \left(\frac{\prod_{i \in I} x_i}{q}\right)^{\deg P}\right)^{(-1)^{|I|-1}}$$

$$= \prod_{I \neq \emptyset: I \subset [2k]} \left(1 - \prod_{i \in I} x_i\right)^{(-1)^{|I|-1}},$$

which is how \(c(m, k)\) was defined!

We have therefore proved that if \(n \geq 2mk\) then

\[\text{Poly}_q(n, m; k) = c(m, k)(1 + O_k(1/q)).\]
Theorems

We have if $N \geq 2mk$ then

$$\text{Perm}(N, m; k) = c(m, k),$$

where

$$c(m, k) \leq m^{2^{2k-1}-2^k-1} + 1.$$

If $n \geq 2mk$ then

$$\text{Poly}_q(n, m; k) = c(m, k)(1 + O_k(1/q)).$$

What is the exponent for the question in the integers?
Sieve weights

Approaching the number theory problem

For $M(n; R) = \sum_{d|n, \, d \leq R} \mu(d)$, as $x/R^{2k} \to \infty$,

$$\frac{1}{x} \sum_{n \leq x} M(n; R)^{2k} \sim \sum_{d_1, \ldots, d_k \leq R} \frac{\mu(d_1) \cdots \mu(d_{2k})}{[d_1, \ldots, d_{2k}]}.$$

Lcm a pain. Break up $d_i$’s into gcds: Given $d_i$’s define $d_I$ for non-empty $I \subset [2k]$, by induction on $2k - |I|:

$$d_{[2k]} = \gcd(d_1, \ldots, d_{2k}).$$

Otherwise

$$d_I = \gcd(d_i : i \in I)/ \prod_{I \subsetneq J \subset [2k]} d_J.$$

Then $d_I$ are pairwise coprime (as $d_i$ are squarefree) and each

$$d_i = \prod_{I: i \in I} d_I.$$
Approaching our problem, 2

\[ \frac{1}{x} \sum_{n \leq x} M(n; R)^{2k} \sim \sum_{d_1, \ldots, d_k \leq R} \mu(d_1) \cdots \mu(d_{2k}) \]

\[ \left[ d_1, \ldots, d_{2k} \right] \]

\[ \mu \left( d_I \right) |I| \]

\[ \frac{d_I}{d_I} \]

where \( d_I \) are pairwise coprime and \( d_i = \prod_{I: i \in I} d_I \leq R \).

Use Perron to get (changing \( d_I \) to \( m_I \))

\[ \frac{1}{(2i\pi)^{2k}} \int \cdots \int \prod_{I \subseteq [2k]} \sum_{m_I \geq 1} \frac{\mu(m_I) |I|}{m_I^{1+s_I}} \prod_j \frac{R^{s_j}}{s_j} ds_j \]

where \( s_I := \sum_{i \in I} s_i \). We see \( \zeta(1+s_I)^{\pm 1} \) factors:

“+” if \(|I|\) is even, “-” if \(|I|\) is odd.
Being precise about the zeta-function

\[ \prod_{\emptyset \neq I \subset [2k]} \sum_{m_I \geq 1} \frac{\mu(m_I)|I|}{m_I^{1+s_I}} \]

\[ = \prod_{p \text{ prime}} \left( 1 + \sum_{\emptyset \neq I \subset [2k]} \frac{(-1)^{|I|}}{p^{1+s_I}} \right). \]

The factor at each prime \( p \) therefore looks like

\[ = \prod_{\emptyset \neq I \subset [2k]} \left( 1 - \frac{1}{p^{1+s_I}} \right)^{-(1)^{|I|}} \]

plus terms \( 1/p^2 + \ldots \). Hence our zeta-function equals

\[ \frac{\prod_{|I| \text{ even}} \zeta(1 + s_I)}{\prod_{|I| \text{ odd}} \zeta(1 + s_I)} \cdot E(s_1, \ldots, s_{2k}) \]

where \( E(s_1, \ldots, s_{2k}) \) is abs cvgent if each \( \text{Re}(s_i) > \epsilon \).
Approaching our problem, 3

Ignore gcd of $M_I$’s (2ndary terms); integral becomes

$$
\frac{1}{(2i\pi)^{2k}} \int \cdots \int \prod_{|I| \text{ even}} \zeta(1 + s_I) \frac{\prod_{|I| \text{ odd}} \zeta(1 + s_I)}{R^{s[k]} \prod_j ds_j} \frac{R^{s[k]} \prod_j ds_j}{s_j}
$$

where the products are over non-empty subsets $I$ of $[2k]$. The plan is to move contours to the left and pick up the $(2k$-dimensional) poles. If some $s_j = 0$ at the pole then:

$$
\zeta(1 + s_I)/\zeta(1 + s_{I \cup \{j\}}) \rightarrow 1 \text{ whenever } j \notin I, \text{ and }
\zeta(1 + s_I)/\zeta(1 + s_j) \rightarrow 1, \text{ as } s_j \rightarrow 0, \text{ so no contribution.}
$$

The integrand is now $\prod_{i \neq j} R^{s_i} / s_i$, and so the pole must be at $(0, 0, \ldots, 0)$, a multi-pole of order $2k - 1$.

(Power of $\log R) = (\text{Order of Pole}) - 2k = -1.$

So this contributes a $c/\log R$ term. Where else?
Approaching our problem, 4

Need to find highest order pole of
\[
\frac{\prod_{|I| \text{ even}} \zeta(1 + s_I)}{\prod_{|I| \text{ odd}} \zeta(1 + s_I)} \cdot \frac{R^{s[2k]}}{s_1 \cdots s_{2k}}.
\]

with all \( s_i \neq 0 \). Order of pole equals
\[
\#\{I \subset [2k], |I| \text{ even} : s_I = 0\} - \#\{I \subset [2k] |I| \text{ odd} : s_I = 0\} - 1.
\]

Maximum is achieved if and only if \( s_i = s(\neq 0) \) for \( k \) values of \( i \), and \( s_j = -s \) for the other \( k \) values of \( j \). Then the power of \( \log R \) is
\[
\binom{2k}{k} - 0 - 1 - (2k - 1) = \binom{2k}{k} - 2k
\]
(the poles come from shifting all but \( s_{2k} \)).
Number theoretic results

One can use this to prove

$$\frac{1}{x} \sum_{n \leq x} M(n; R)^{2k} \sim c_k (\log R)^{\binom{2k}{k} - 2k}$$

where $c_k$ is some non-zero constant. That is,

$$\text{Exponent}(k) = \binom{2k}{k} - 2k.$$

More generally:

**Theorem 2.** Let

$$f_A(t) := \begin{cases} (1 - t)^A & \text{for } t \leq 1; \\ 0 & \text{otherwise,} \end{cases}$$

which is $A$-times differentiable on $\mathbb{R}$. Then

$$\frac{1}{x} \sum_{n \leq x} M_{f_A}(n; R)^{2k} \sim c_{k,A} (\log R)^{\binom{2k}{k} - 2k(A+1)} + \frac{c_{k,A}}{\log R},$$

where $c_{k,A}$ is some non-zero constant.
Comparing results for Möbius moments

For the permutation moments, and the polynomials in finite field moments, the exponent is

\[ 2^{2k-1} - 2k - 1. \]

For the integer moments, the exponent is

\[ \binom{2k}{k} - 2k. \]

???
Comparing results for Möbius moments

For the permutation moments, and the polynomials in finite field moments, the exponent is

\[ 2^{2k-1} - 2k - 1. \]

For the integer moments, the exponent is

\[ \binom{2k}{k} - 2k. \]

Much seems to depend on whether \( \zeta \)-poles in the denominator of the zeta-function cancel those in the numerator. For quadratic \( \chi \mod q \), try

\[
\frac{1}{x} \sum_{n \leq x} \left| \sum_{d | n, \ d \leq R} \chi(d) \right|^{2k}
\]
Quadratic character $\chi \pmod{q}$, moments

$$\frac{1}{x} \sum_{n \leq x} \left| \sum_{d|n, \ d \leq R} \chi(d) \right|^{2k}$$

Integrand product – look for poles in

$$\prod_{|I| \text{ even, } > 0} L(1 + s_I, \chi_0) \cdot \prod_{|I| \text{ odd}} L(1 + s_I, \chi).$$

Pole at $(0, \ldots, 0)$ order $2^{2k-1} - 1$. 
Sieve weights

**Quadratic character** $\chi \pmod{q}$, moments

\[
\frac{1}{x} \sum_{n \leq x} \left| \sum_{d|n, \ d \leq R} \chi(d) \right|^{2k}
\]

Integrand product – look for poles in

\[
\prod_{|I| \text{ even}, > 0} L(1 + s_I, \chi_0) \cdot \prod_{|I| \text{ odd}} L(1 + s_I, \chi).
\]

Pole at $(0, \ldots, 0)$ order $2^{2k-1} - 1$.

If $\chi$ has a Siegel zero then $L(1 + s_I, \chi) \approx \zeta(1 + s_I)^{-1}$ in a limited range. Asymptotic:

\[
c_1 \ L(1, \chi)^{2^{2k-1}} \left( \frac{\phi(q)}{q} \log R \right)^{2^{2k-1} - 1} + c_2(q)(\log R)^{(2k^k) - 2k}.
\]