Modelling large values of L-functions

How big can things get?

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How big can the Riemann zeta function get?

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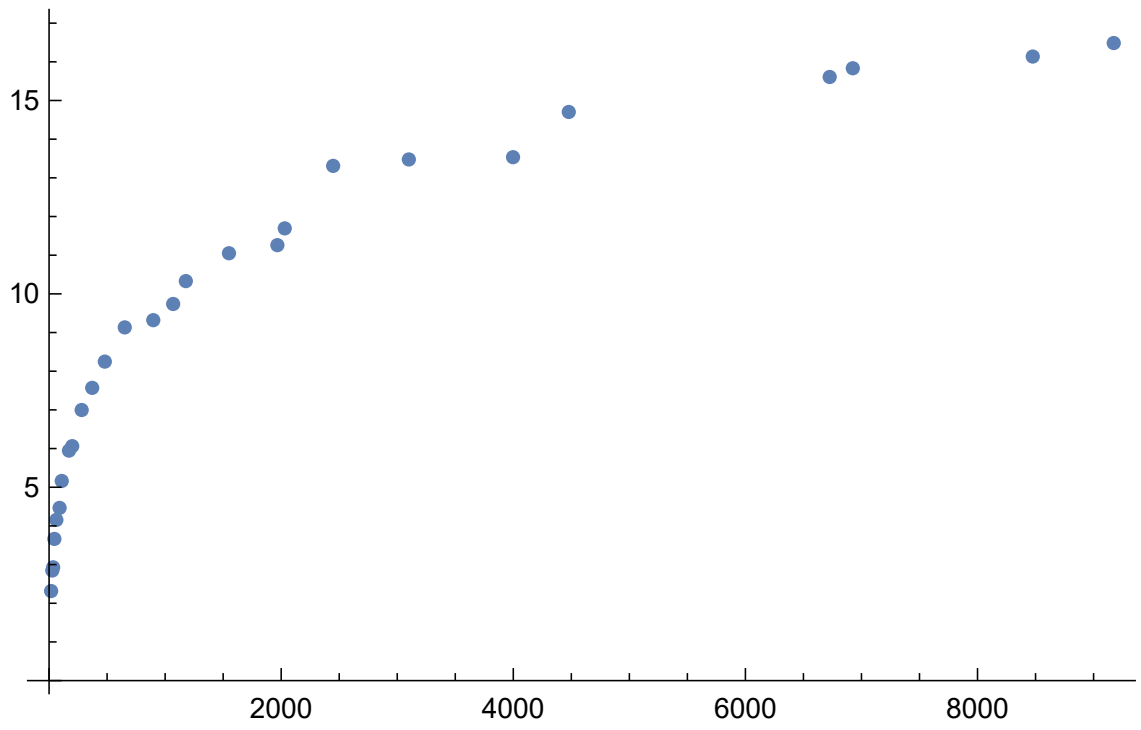
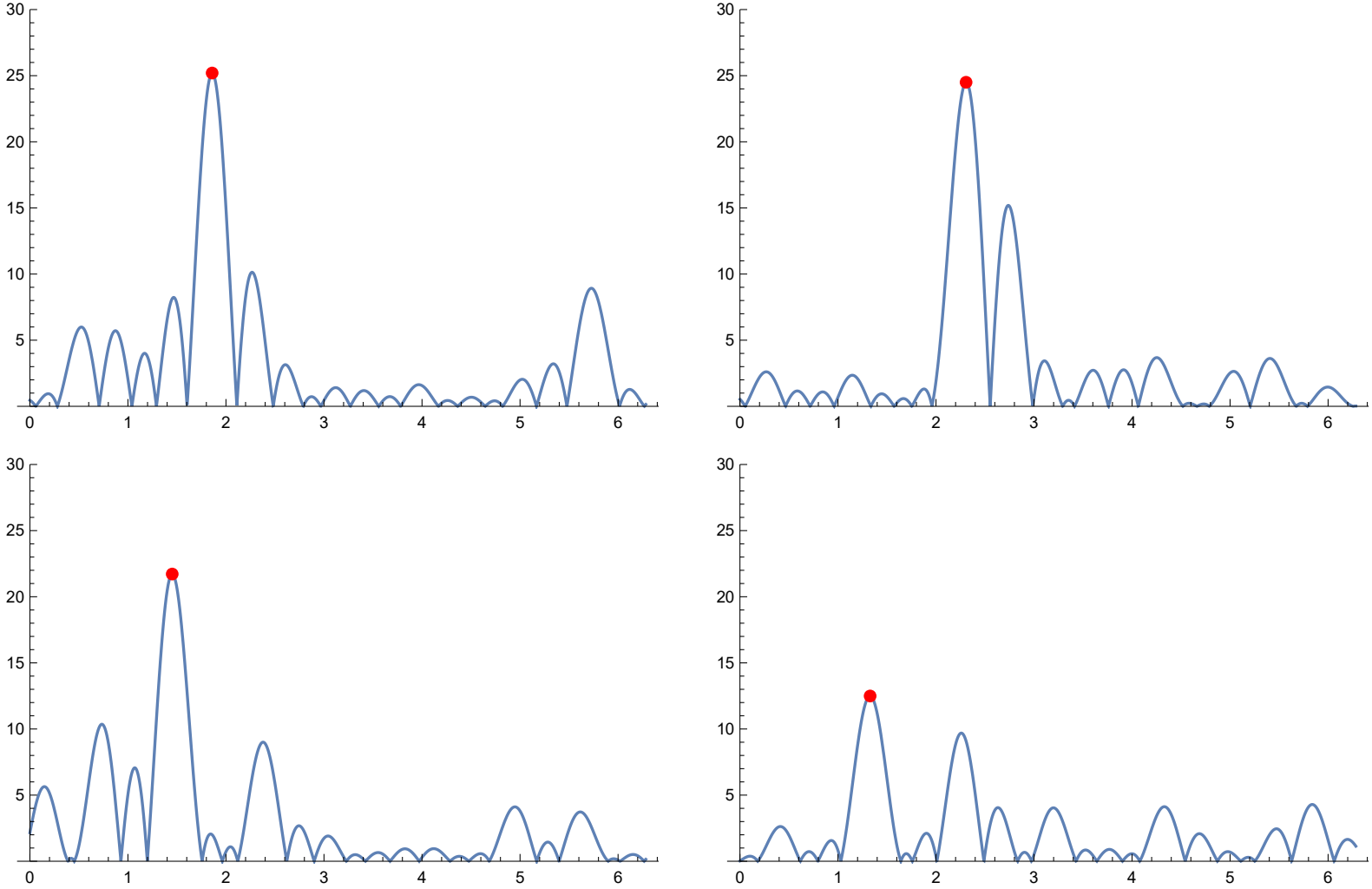
$$\zeta$$

Extreme values of the Riemann zeta function



The running maxima of zeta
for $0 \leq t \leq 10^3$

Large values of the Riemann zeta function



The largest value of zeta over an interval of length 2π for $t = 10^{10}, 10^{10} + 100, 10^{10} + 200, 10^{10} + 300$

Extreme values of zeta

(Growth of maxima up to height T)

Extreme values of zeta

Conjecture (Farmer, Gonek, Hughes)

$$\max_{t\in[0,T]} |\zeta(\tfrac{1}{2}+\mathrm{i}t)| = \exp\left(\left(\frac{1}{\sqrt{2}}+o(1)\right)\sqrt{\log T \log\log T}\right)$$

Bounds on extreme values of zeta

Theorem (Littlewood; Ramachandra and Sankaranarayanan; Soundararajan; Chandee and Soundararajan)

Under RH, there exists a C such that

$$\max_{t\in[0,T]} |\zeta(\tfrac{1}{2}+\mathrm{i}t)| = O\left(\exp\left(C\frac{\log T}{\log\log T}\right)\right)$$

Theorem (Bondarenko-Seip)

For all $c < 1/\sqrt{2}$

$$\max_{t\in[0,T]} |\zeta(\tfrac{1}{2}+\mathrm{i}t)| > \exp\left(c\sqrt{\frac{\log T \log\log\log T}{\log\log T}}\right)$$

An Euler-Hadamard hybrid

Theorem (Gonek, Hughes, Keating)

A simplified form of our theorem is:

$$\zeta(\tfrac{1}{2}+\mathrm{i}t) = P(t;X)Z(t;X) + \textit{errors}$$

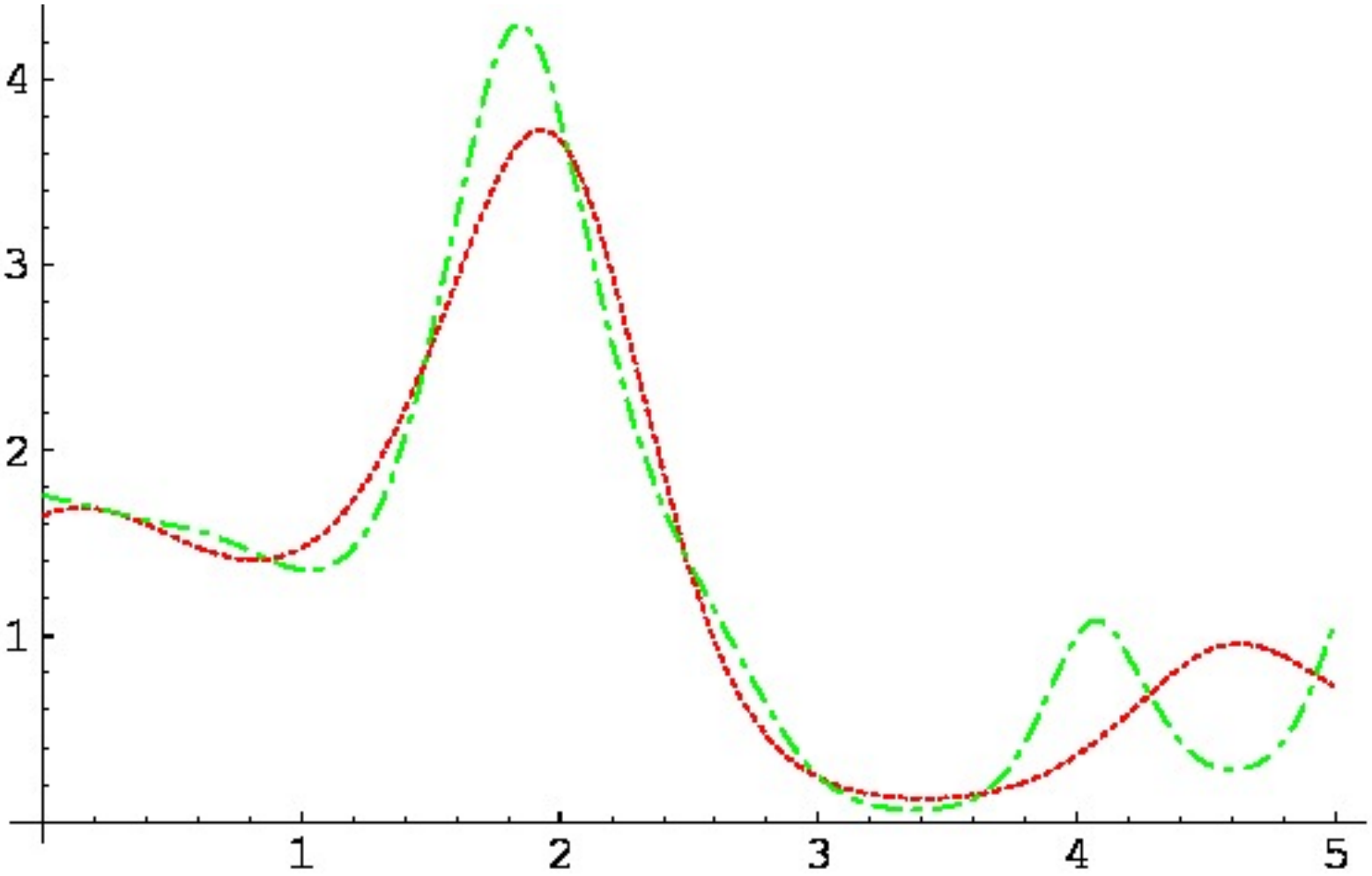
where

$$P(t;X) = \prod_{p\leq X}\left(1-\frac{1}{p^{\frac{1}{2}+\mathrm{i}t}}\right)^{-1}$$

and

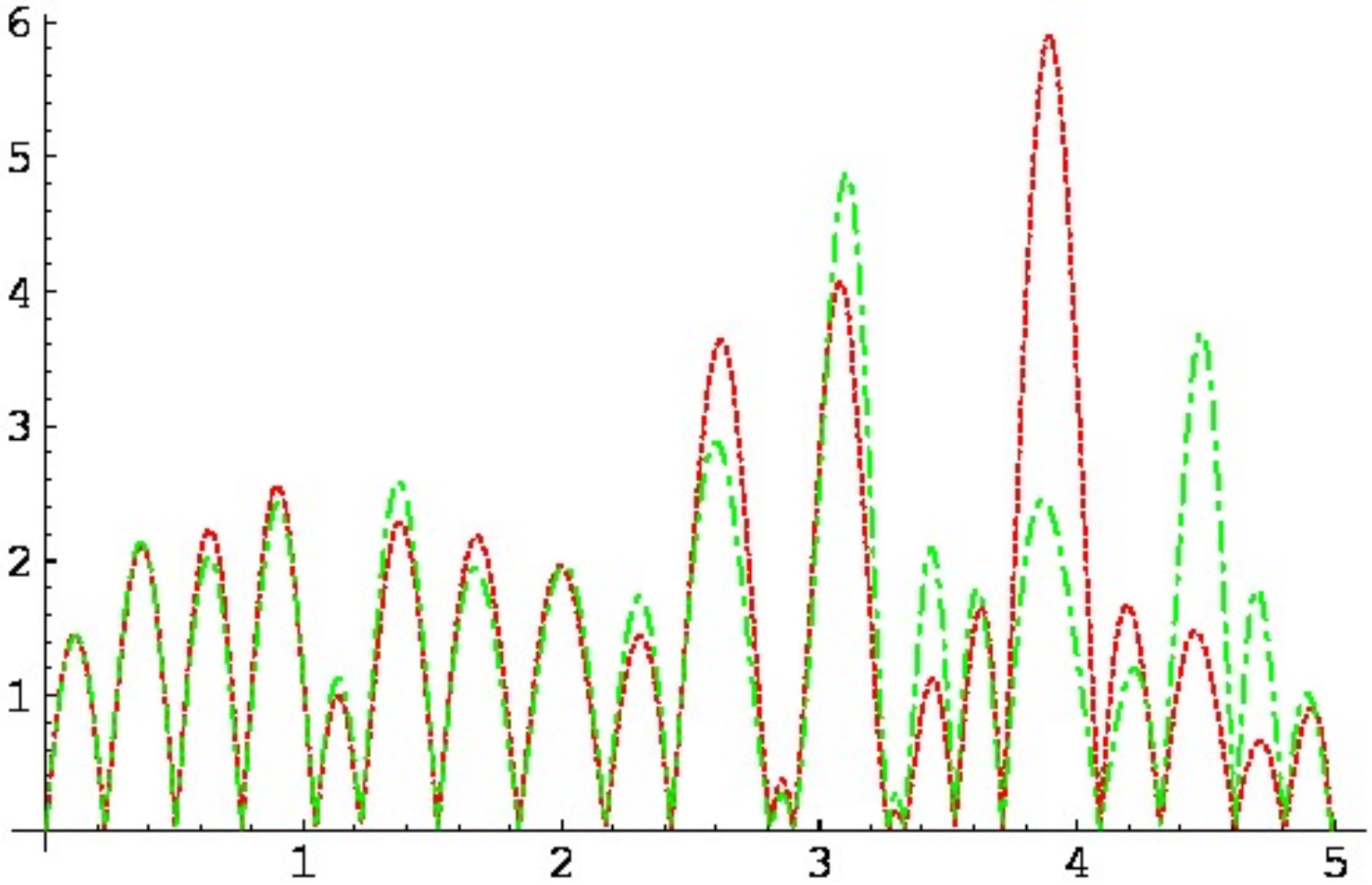
$$Z(t;X) = \exp\left(\sum_{\gamma_n}\mathrm{Ci}(|t-\gamma_n|\log X)\right)$$

An Euler-Hadamard hybrid: Primes only



Graph of $|P(t+t_0;X)|$, with $t_0 = \gamma_{10^{12}+40}$,
with $X = \log t_0 \approx 26$ (red) and $X = 1000$ (green).

An Euler-Hadamard hybrid: Zeros only



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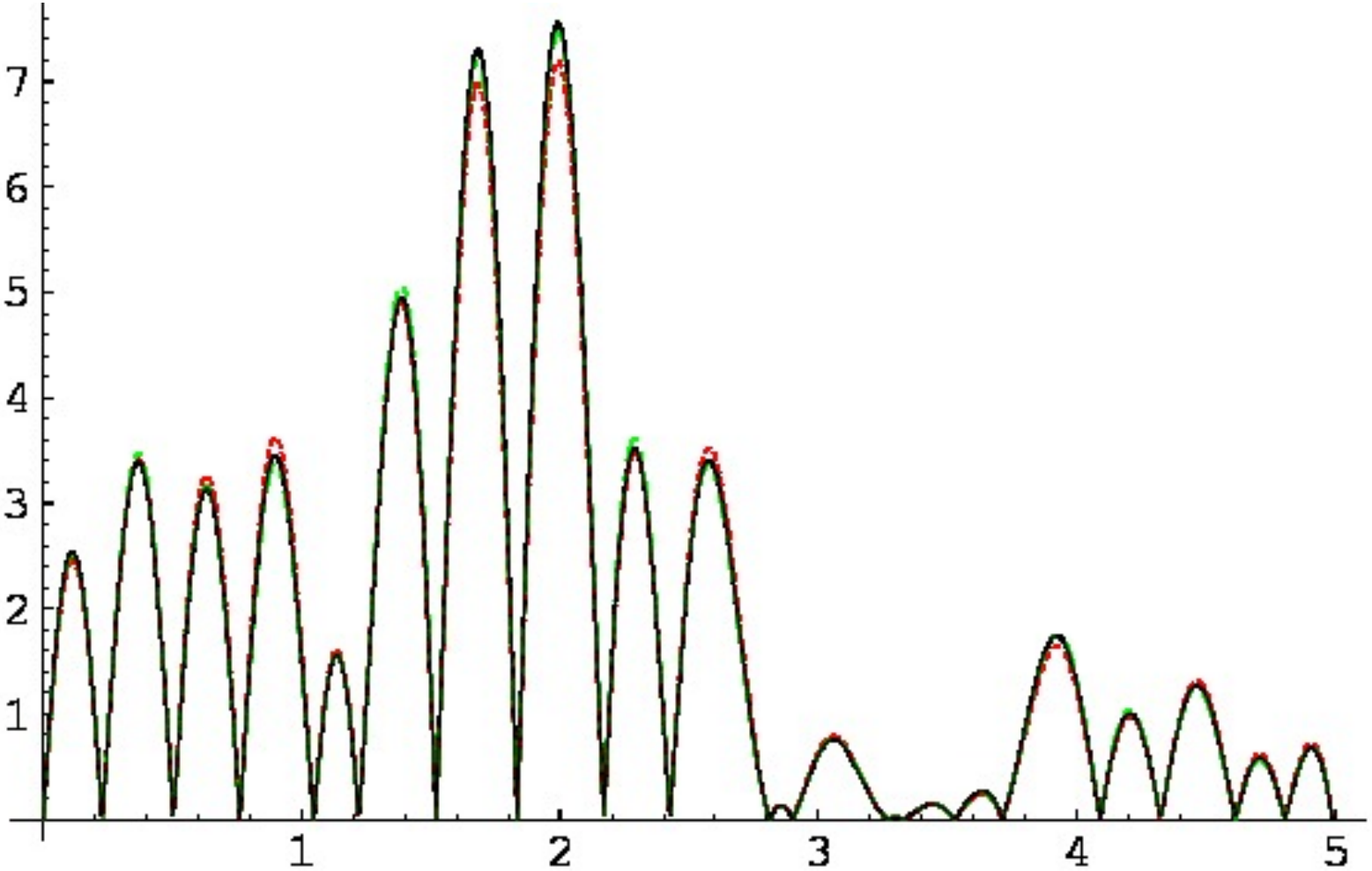
An Euler-Hadamard hybrid: Primes and zeros



Graph of $|\zeta(\frac{1}{2} + \mathrm{i}(t + t_0))|$ (black) and $|P(t + t_0; X)Z(t + t_0; X)|$, with $t_0 = \gamma_{10^{12}+40}$, with $X = \log t_0 \approx 26$ (red) and $X = 1000$ (green).

Extreme values of zeta: RMT & zeros

Keating and Snaith modelled the Riemann zeta function with

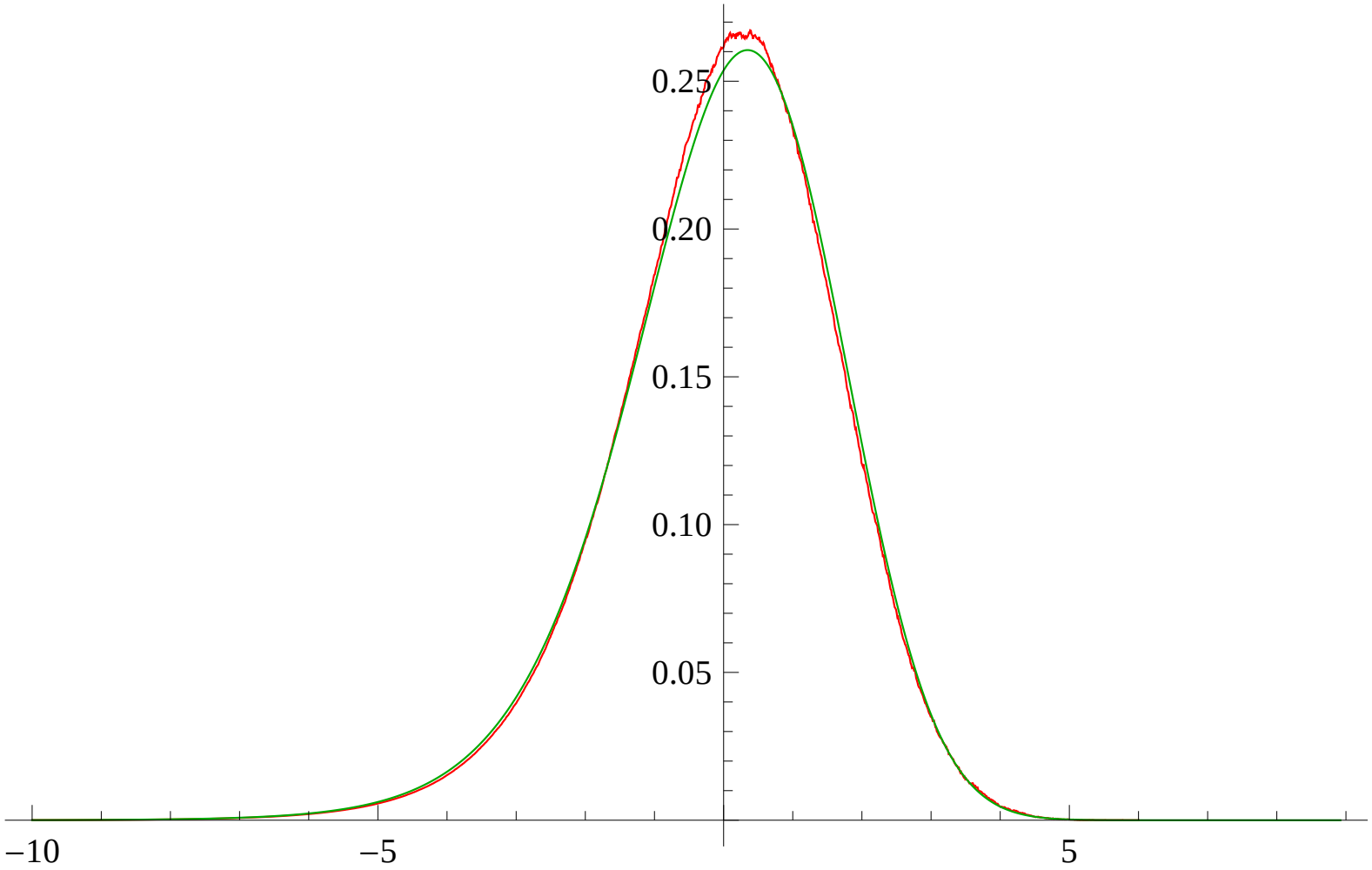
$$Z_{U_N}(\theta) := \det(I_N - U_N e^{-\mathrm{i}\theta})$$
$$= \prod_{n=1}^{N} (1 - e^{\mathrm{i}(\theta_n - \theta)})$$

where U_N is an $N \times N$ unitary matrix chosen with Haar measure.

The matrix size N is connected to the height up the critical line T via

$$N = \log \frac{T}{2\pi}$$

Extreme values of zeta: RMT & zeros



Graph of the value distribution of $\log|\zeta(\frac{1}{2}+\mathrm{i}t)|$ around the 10^{20}th zero (red), against the probability density of $\log|Z_{U_N}(0)|$ with $N=42$ (green).

Extreme values of zeta: RMT & zeros

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Model each block with the characteristic polynomial of an $N \times N$ random unitary matrix.

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blocks, each containing approximately N zeros.
Model each block with the characteristic polynomial of an $N \times N$ random unitary matrix.
Find the smallest $K = K(M, N)$ such that choosing M independent characteristic polynomials of size N, almost certainly none of them will be bigger than K.

Extreme values of zeta: RMT & zeros

Note that

$$\mathbb{P}\left\{\max_{1\le j\le M}\max_{\theta}|Z_{U_N^{(j)}}(\theta)|\le K\right\}=\mathbb{P}\left\{\max_{\theta}|Z_{U_N}(\theta)|\le K\right\}^M$$

Theorem

Let $0<\beta<2$. *If* $M=\exp(N^\beta)$, *and if*

$$K=\exp\left(\sqrt{(1-\tfrac{1}{2}\beta+\varepsilon)\log M\log N}\right)$$

then

$$\mathbb{P}\left\{\max_{1\le j\le M}\max_{\theta}|Z_{U_N^{(j)}}(\theta)|\le K\right\}\to 1$$

as $N\to\infty$ *for all* $\varepsilon>0$, *but for no* $\varepsilon<0$.

Extreme values of zeta: RMT & zeros

Recall

$$\zeta(\tfrac{1}{2} + \mathrm{i}t) = P(t; X)Z(t; X) + \text{errors}$$

We showed that $Z(t; X)$ can be modelled by characteristic polynomials of size

$$N = \frac{\log T}{e^{\gamma} \log X}$$

Extreme values of zeta: RMT & zeros

Recall

$$\zeta(\tfrac{1}{2}+\mathrm{i}t) = P(t;X)Z(t;X) + \text{errors}$$

We showed that $Z(t;X)$ can be modelled by characteristic polynomials of size

$$N = \frac{\log T}{e^{\gamma}\log X}$$

Therefore the previous theorem suggests

Conjecture

If $X = \log T$*, then*

$$\max_{t\in[0,T]} |Z(t;X)| = \exp\left(\left(\frac{1}{\sqrt{2}} + o(1)\right)\sqrt{\log T \log\log T}\right)$$

Extreme values of zeta: RMT & zeros

Theorem

By the PNT, if $X = \log T$ *then for any* $t \in [0, T]$,

$$P(t; X) = O\left(\exp\left(C\frac{\sqrt{\log T}}{\log\log T}\right)\right)$$

Thus one is led to the max values conjecture

Conjecture

$$\max_{t\in[0,T]} |\zeta(\tfrac{1}{2} + \mathrm{i}t)| = \exp\left(\left(\frac{1}{\sqrt{2}} + o(1)\right)\sqrt{\log T \log\log T}\right)$$

Extreme values of zeta: Primes

First note that

$$P(t;X) = \exp\left(\sum_{p\leq X} \frac{1}{p^{1/2+\mathrm{i}t}}\right) \times O(\log X)$$

Extreme values of zeta: Primes

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Treat $p^{-\mathrm{i}t}$ as independent random variables, U_p, distributed uniformly on the unit circle.
The distribution of

$$\mathfrak{Re}\sum_{p\le X}\frac{U_p}{\sqrt{p}}$$

tends to Gaussian with mean 0 and variance $\frac{1}{2}\log\log X$ as $X\to\infty$.

Extreme values of zeta: Primes

We let $X = \exp(\sqrt{\log T})$ and model the maximum of $P(t; X)$ by finding the maximum of the Gaussian random variable sampled $T(\log T)^{1/2}$ times. This suggests

$$\max_{t \in [0,T]} |P(t; X)| = O\left(\exp\left((\frac{1}{\sqrt{2}} + \varepsilon)\sqrt{\log T \log\log T}\right)\right)$$

for all $\varepsilon > 0$ and no $\varepsilon < 0$.

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for all $\varepsilon > 0$ and no $\varepsilon < 0$.
For such a large X, random matrix theory suggests that

$$\max_{t \in [0,T]} |Z(t; X)| = O\left(\exp\left(\sqrt{\log T}\right)\right).$$

This gives another justification of the large values conjecture.

Large values of zeta

(Distribution of maxima over short intervals)

Distribution of the max of characteristic polynomials

In 2012 Fyodorov, Hiary and Keating studied the distribution of the maximum of a characteristic polynomial of a random unitary matrix via freezing transitions in certain disordered landscapes with logarithmic correlations. This mixture of rigorous and heuristic calculation led to:

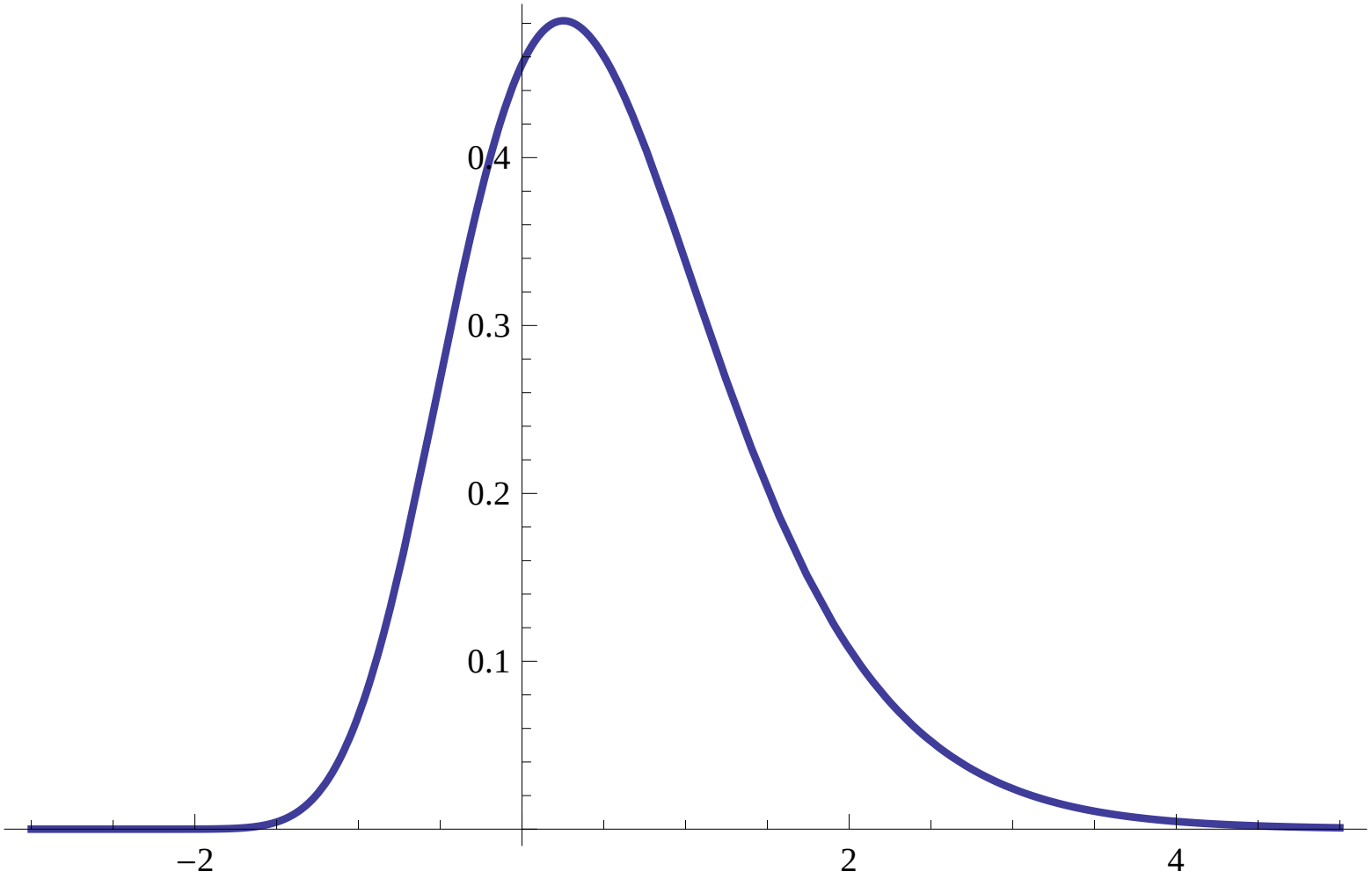
Conjecture (Fyodorov, Hiary and Keating)

For large N,

$$\log \max_{\theta} |Z_{U_N}(\theta)| \sim \log N - \frac{3}{4} \log \log N + Y$$

where the random variable Y has the density $4e^{-2y} K_0(2e^{-y})$

Distribution of the max of characteristic polynomials



The probability density of Y

Distribution of the max of characteristic polynomials

Note that

$$\mathbb{P}\{Y \geq K\} \approx 2Ke^{-2K}$$

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However, one can show that if $K/\log N \to \infty$ but $K \ll N^{\epsilon}$ then

$$\mathbb{P}\left\{\max_{\theta} \log |Z_{U_N}(\theta)| \geq K\right\} = \exp\left(-\frac{K^2}{\log N}(1 + o(1))\right)$$

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Thus there must be a critical K (of the order $\log N$) where the probability that $\max_{\theta} |Z_U(\theta)| \approx K$ changes from looking like linear exponential decay to quadratic exponential decay.

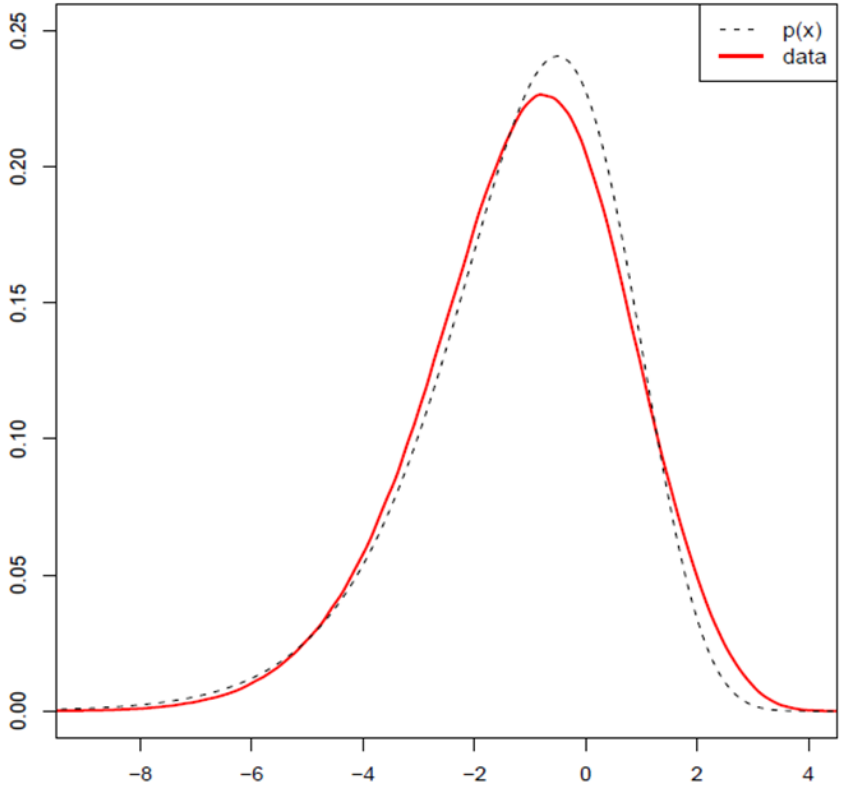
Large values of zeta

Their work with characteristic polynomials led Fyodorov and Keating to conjecture that

$$\max_{T \le t \le T+2\pi} |\zeta(\tfrac{1}{2} + \mathrm{i}t)| \sim \exp\left(\log\log\left(\frac{T}{2\pi}\right) - \frac{3}{4}\log\log\log\left(\frac{T}{2\pi}\right) + Y\right)$$

with Y having (approximately) the same distribution as before.

Large values of zeta



Distribution of $-2\log\max_{t\in[T,T+2\pi]}|\zeta(\frac{1}{2}+\mathrm{i}t)|$ (after rescaling to get the empirical variance to agree) based on 2.5×10^8 zeros near $T=10^{28}$. Graph by Ghaith Hiary, taken from Fyodorov-Keating.

Large values of zeta

The conjecture that for almost all T

$$\max_{0\le h\le 1} \log|\zeta(\tfrac{1}{2}+\mathrm{i}(T+h))| = \log\log T - \frac{3}{4}\log\log\log T + O(1)$$

was backed up by a different argument of Harper, and later by Arguin, Belius and Harper.
They considered random Euler products

$$\max_{0\le h\le 1} \sum_{p\le T} \frac{\mathfrak{Re}(U_p p^{-\mathrm{i}h})}{\sqrt{p}}$$

where U_p are uniform iid on the unit circle.

Large values of zeta

Split the sum over primes up

$$Y_k(h) = \sum_{2^{k-1} < \log p \leq 2^k} \frac{\mathfrak{Re}(U_p p^{-\mathrm{i}h})}{\sqrt{p}}$$

These are well-correlated if $|h - h'| < 2^{-k}$ and nearly uncorrelated for h and h' further apart.
Translating this as a branching random walk they were able to show that

$$\max_{0 \leq h \leq 1} \sum_{p \leq T} \frac{\mathfrak{Re}(U_p p^{-\mathrm{i}h})}{\sqrt{p}} = \log\log T - \frac{3}{4}\log\log\log T + o_P(\log\log\log T)$$

Moments at the local maxima

Moments of the local maxima

Theorem (Conrey and Ghosh)

As $T \to \infty$

$$\frac{1}{N(T)} \sum_{t_n \leq T} \left|\zeta(\tfrac{1}{2} + \mathrm{i}t_n)\right|^2 \sim \frac{e^2 - 5}{2} \log T$$

where t_n are the points of local maxima of $|\zeta(\frac{1}{2} + \mathrm{i}t)|$.

This should be compared with Hardy and Littlewood's result

$$\frac{1}{T} \int_0^T |\zeta(\tfrac{1}{2} + \mathrm{i}t)|^2 \mathrm{d}t \sim \log T$$

Moments of the local maxima

In 2012 Winn succeeding in proving a random matrix version of this result (in disguised form)

Theorem (Winn)

As $N \to \infty$

$$\mathbb{E}\left[\frac{1}{N}\sum_{n=1}^{N}\left|Z_{U_N}(\phi_n)\right|^{2k}\right] \sim C(k)\,\mathbb{E}\left[\left|Z_{U_N}(0)\right|^{2k}\right]$$

where ϕ_n are the points of local maxima of $\left|Z_{U_N}(\theta)\right|$, and where $C(k)$ can be given explicitly as a combinatorial sum involving Pochhammer symbols on partitions.

In particular,

$$\mathbb{E}\left[\frac{1}{N}\sum_{n=1}^{N}\left|Z_{U_N}(\phi_n)\right|^{2}\right] \sim \frac{e^2-5}{2}N$$

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