RESTRICTION OF BANACH REPRESENTATIONS OF $GL_2(\mathbb{Q}_p)$ TO $GL_2(\mathbb{Z}_p)$

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ABSTRACT. Thanks to the p-adic local Langlands correspondence for $GL_2(\mathbb{Q}_p)$, one "knows" all admissible unitary topologically irreducible representations of $GL_2(\mathbb{Z}_p)$. In this talk I will focus on some elementary properties of their restriction to $GL_2(\mathbb{Z}_p)$: for instance, to what extent does the restriction to $GL_2(\mathbb{Z}_p)$ allow one to recover the original representation, when is the restriction of finite length, etc.

Let p be a prime number and

$$G = GL_2(\mathbb{Q}_p) \supset K = GL_2(\mathbb{Z}_p). \tag{0.1}$$

Schneider and Teitlebaum: introduce an abelian category of admissable Banach space representations of G or K (or any p-adic Lie group).

To reflect on the GL_2-side, the existence of Galois-stable lattices in representations, I will deal only with unitary representations of G. I will consider $\mathbb{Q}_p$-Banach spaces $\Pi + \mathbb{Q}_p$ linear action of G by unitary transformations. ($\exists$ G-invariant norm on Π defining its topology + finiteness).

If $\Pi_0 = \{v \in \Pi : |v| \leq 1\}$, then $\Pi_0/p\Pi_0$ is admissable smooth, i.e., $\forall H \leq K$ compact open subgroups $(\Pi_0/p\Pi_0)^H$ is finite. I will only deal with finite lenght objects in this category $\mathscr{C}$.

Theorem 0.1 (Paskunas)**.** Π *has finite lenght* $\Rightarrow \Pi_0/p\Pi_0$ *has finite lenght.*

Question 0.2. *What about other groups?*

Colmez created a magical exact functor

$$\left\{ \text{ finite lenght objects in } \mathscr{C} \right\} \longrightarrow \left\{ \begin{array}{l} \text{finite dimensional continuous} \\ \mathbb{Q}_p\text{-representation of } \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \end{array} \right\} \tag{0.2}$$

Problem for Today: Describe the functor $\Pi \to \Pi/K$ in terms of Galois representations. (Caraiani, Emerton, Gee, Geraghty, Paskunas, Shin).

Question 0.3. *Is it true that this operation preserves finite lenght?*

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All errors are the responsibility of the typesetter. In particular there are some arguments which, as an exercise for the typesetter, have been fleshed out or reinterpreted, possibly incorrectly. The lecture is always neater and more concise.

Let π be an admissable smooth infinite dimensional irreducible representation of G

$$\pi|_K = \bigoplus_{i\geq 1} \pi_i, \qquad \pi_i \text{ is } K\text{-invariant.} \tag{0.3}$$

If $\pi \hookrightarrow \Pi$, then $\Pi|_K$ must have infinite lenght. More generally, if Π contains locally $SL_2(\mathbb{Q}_p)$-algebraic vectors ($v \in \Pi$ whose orbit map $SL_2(\mathbb{Q}_p)$) then Π is locally polynomial$\Rightarrow$ $\Pi|_K$ must be of infinite lenght.

Theorem 0.4 (Colmez)**.** *This happens if and only if $V(\Pi)$ (Galois representation) is de Rham with $\neq$ Hodge-Tate weights up to a twist.*

Conjecture 0.5. *These are precisely the exceptions to the statement: $\Pi \in \mathscr{C}$ irreducible $\Rightarrow \Pi/K$ finite lenght.*

Remark.

Theorem 0.6 (Colmez, Paskunas, D., Berger-Breuil)**.** *Colmez functor gives a bijection*

$$\left\{ \begin{array}{l} \textit{irreducible objects in } \mathscr{C} \\ \textit{which are not subquotients} \\ \textit{parabolic inductions} \end{array} \right\} \Big/ \simeq \longrightarrow \left\{ \begin{array}{l} \textit{irreducible 2-dimensiona} \\ \textit{representation of } Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \\ \textit{over } \mathbb{Q}_p \end{array} \right\} \Big/ \simeq \tag{0.4}$$

Theorem 0.7. *Say that $\Pi|_K$ has ∞ lenght then $V(\Pi)$ is Hodge-Tate with distinct weights up to a twist. Moreover, in this case there is a finite dimensional K-subquotient in Π.*

Remark. *If one proves that the existence of such subquotients forces the existence of subobjects, one would get the conjecture.*

Question 0.8. *Is it true that*

$$H^1(K, \sigma \otimes W) = 0 \tag{0.5}$$

for any subobject σ of Π and any finite dimensional representation W of K?

Question 0.9. *What about full faithfulness of $\Pi \to \Pi|_K$?*

Answer: No chance, but almost!

Theorem 0.10 (Caraiani, Emerton, Gee, Geraghty, Paskunas, Shin)**.** *If $\Pi_1, \Pi_2 \in \mathscr{C}$, then*

$$\mathrm{Hom}_K(\Pi_1, \Pi_2) = \mathrm{Hom}_{\mathrm{inertia}}(V(\Pi_1), V(\Pi_2)). \tag{0.6}$$

Remark. *If $V(\Pi)|_{\mathrm{inertia}}$ is irreducible then $End_K(\Pi)$ is a division algebra.*

Hence, if $\Pi|_K$ has finite lenght, then it is probably irreducible.

0.1. **Ideas giving the proof of Theorem 0.7.** Key point: combine non-commutative algebra and what we know about the reduction modulo p.

$$\Pi \in \mathscr{C} \rightsquigarrow \Pi^* = \mathrm{Hom}^{\text{cont.}}_{\mathbb{Q}_p}(\Pi, \mathbb{Q}_p), \tag{0.7}$$

a module over $\Lambda(K) = \mathbb{Z}_p[[K]] \otimes \mathbb{Q}_p$. Admissibility implies Π^* is finitely generated over this ring.

Replacing K with a smaller subgroup we get a finite module over an Auslander regular ring of global dimension 4.

The canonical dimension of any finite module over it is between 0 and 4. (0 if and only if is of finite dimension over $\mathbb{Q}_p$). If $M = M_0 \supset M_1 \supset \cdots$ decreasing sequence of submodules then for $n \gg 0$, $\dim(M_n/M_{n+1}) < \dim(M)$.

Hard part: $\dim(\Pi^*) = 1$.

Work of Ardakov + existence of infinitesimal characters $\Rightarrow \dim(\Pi^*) \leq 2$. Going from 2 to 1 is hard.

If one uses Paskunas' results (reduction mod p has finite lenght), one is reduced to the following statement: if π is an irreducible admissable infinite dimensional mod p representation of G (admissable smooth) then

$$\dim_{\mathbb{F}_p} \pi^{1+p^n M_2(\mathbb{Z}_p)} \leq \text{cte}.p^n, \qquad \forall\ n. \tag{0.8}$$

Done by Mora if $p > 2$.