ON ELEMENTARY ESTIMATES OF ARITHMETIC SUMS FOR POLYNOMIAL RINGS OVER FINITE FIELDS

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ABSTRACT. In this paper, a simple and elementary method is given for deriving estimates of sums of arithmetic functions in $\mathbb{F}_q[t]$. The method is the function field analogue of a result first proved by Stefan A. Burr in 1973 in the number field case. A novelty of this paper is that we are able to extend Burr's result, in the function field context, and obtain secondary main terms for the appropriate sums involving the divisor functions $d_r(f)$ with an error term that improves the one given by Burr.

1. INTRODUCTION

Burr [3] gave a variant of a method developed by Bateman [2] and Tull [10, 11] to estimate sums of arithmetic functions by means of convolution arguments. The estimates provided by Burr are generally uniform over a class of functions appearing in the summation and the results prove useful in applications of Selberg's sieve, where the functions appearing tend to be rather complicated.

When applying Selberg's sieve, the required estimates are of a rather special type, meaning that it is useful to have easily applicable and simple estimates. The results in Burr's paper [3] were derived in order to ensure greater accessibility to exploring such applications. It is also important to notice that Burr, in his paper [3], has focused on some simple results that can be greatly generalized.

The main result presented in Burr's paper is given below.

Theorem 1.1 (Burr [3], Theorem 1)**.** *Let f be an arithmetic function and let g be such that*

$$f(n) = \sum_{d|n} g(d)d_r(n/d),$$

with $r \geq 2$ and $d_r(n)$ being the r-fold divisor function. Also let $F(s)$ and $G(s)$ be the Dirichlet series associated to $f(n)$ and $g(n)$ respectively and

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$$G_1(s) = \sum_{n=1}^{\infty} \frac{|g(n)|}{n^s} \qquad \textit{and} \qquad G_2(s) = |g(1)| + \sum_{n=2}^{\infty} \frac{|g(n)| \log n}{n^s}.$$

Suppose that $G_2(1)$ *exists. Then*

$$\sum_{n \le x} f(n) = \frac{1}{(r-1)!} G(1) x (\log x)^{r-1} + O(G_2(1) x (\log x)^{r-2}) \tag{1.1}$$

and

$$\sum_{n \le x} \frac{f(n)}{n} = \frac{1}{r!} G(1)(\log x)^r + O(G_2(1)(\log x)^{r-1}), \tag{1.2}$$

where the implied constants depend only on r*.*

In the first part of this paper (see Section 2) we prove a function field analogue of Theorem 1.1. As an illustration of Burr's results we also prove an estimate involving the divisor function over square-free polynomials that are coprime to a fixed monic polynomial in $\mathbb{F}_q[t]$. This application of Burr's theorem in $\mathbb{F}_q[t]$ is typical when using Selberg's sieve over the ring of polynomials over a finite field. For more details about the Selberg sieve in $\mathbb{F}_q[t]$ see [12]. In the second part of this paper (Section 3) we extend Burr's result, in the function field context, and obtain secondary main terms for the appropriate sums involving the divisor functions $d_r(f)$ with an error term that improves the one given by Burr.

2. Burr's Theorem in Function Fields

Before we state the main result of this section we first introduce some notation and some auxiliary results.

Let $\mathbb{F}_q$ be a finite field with q elements where q is a prime power. We denote $A = \mathbb{F}_q[t]$ the polynomial ring over $\mathbb{F}_q$ and the norm of a polynomial in A is defined to be $|f| = q^{\deg(f)}$.

Let F and G be two arithmetic functions in $\mathbb{F}_q[t]$. We denote by $F^*(s)$, $G^*(s)$, $G_1^*(s)$, and $G_2^*(s)$ the following Dirichlet Series

$$\sum_{f \text{ monic}} \frac{F(f)}{|f|^s}, \tag{2.1}$$

$$\sum_{g \text{ monic}} \frac{G(g)}{|g|^s}, \tag{2.2}$$

$$\sum_{g \text{ monic}} \frac{|G(g)|}{|g|^s}, \tag{2.3}$$

and

$$|G(1)| \ + \sum_{\substack{g \text{ monic} \\ \deg(g)\geq 1}} \frac{|G(g)|\deg(g)}{|g|^s}, \tag{2.4}$$

respectively. Next, let f be a monic polynomial in $\mathbb{F}_q[t]$ and we also define $d_r(f)$ to be the number of ways that f can be expressed as a product of r monics (taking order into account). Note that $\zeta_A^r(s)$ is the Dirichlet series generating function of d_r, where

$$\zeta_A(s) = \sum_{f \text{ monic}} \frac{1}{|f|^s} = \frac{1}{1-q^{1-s}}. \tag{2.5}$$

We will need the results of the following lemma.

Lemma 2.1. *Let $d_r(f)$ defined as above. Then we have the following:*

$$\sum_{\substack{f \text{ monic} \\ \deg(f)=n}} d_r(f) = q^n \binom{n+r-1}{r-1}, \tag{2.6}$$

and

$$\sum_{\substack{f \text{ monic} \\ \deg(f)=n}} d_r(f) = \frac{1}{(r-1)!} q^n n^{r-1} \ + O(q^n n^{r-2}). \tag{2.7}$$

Moreover, we have

$$\sum_{\substack{f \text{ monic} \\ \deg(f)=n}} \frac{d_r(f)}{|f|} = \frac{1}{(r-1)!} n^{r-1} \ + O(n^{r-2}). \tag{2.8}$$

Proof. See Lemma 2.2 in [1] for a proof of (2.6). The equations (2.7) and (2.8) follows easily from (2.6). □

We now state the main result of this section which can be seen as the function field analogue of Burr's result.

Theorem 2.1. *Let F be an arithmetic function in $\mathbb{F}_q[t]$ and let G be such that*

$$F(f) = \sum_{\substack{h \text{ monic} \\ h|f}} G(h) d_r(f/h) \tag{2.9}$$

with $r \geq 2$ and suppose $G_2^*(1)$ exists. Then,

$$\sum_{\substack{f \text{ monic} \\ deg(f)=n}} F(f) = \frac{1}{(r-1)!} G^*(1) q^n n^{r-1} + O(G_2^*(1) q^n n^{r-2}) \tag{2.10}$$

and

$$\sum_{\substack{f \text{ monic} \\ deg(f)=n}} \frac{F(f)}{|f|} = \frac{1}{(r-1)!} G^*(1) n^{r-1} + O(G_2^*(1) n^{r-2}) \tag{2.11}$$

where the implied constants depend only on r.

Proof. We prove (2.10) first. Making use of (2.9) and (2.7), we have

$$\begin{aligned}
\sum_{\substack{f \text{ monic} \\ \deg(f)=n}} F(f) &= \sum_{\substack{h,j \text{ monic} \\ \deg(hj)=n}} G(h) d_r(j) \\
&= \sum_{\substack{h \text{ monic} \\ \deg(h)\leq n}} G(h) \sum_{\substack{j \text{ monic} \\ \deg(j)=n-\deg(h)}} d_r(j) \qquad (2.12) \\
&= \sum_{\substack{h \text{ monic} \\ \deg(h)\leq n}} G(h) \left(\frac{1}{(r-1)!} \frac{q^n}{|h|} (n-\deg(h))^{r-1} + O\Big(\frac{q^n}{|h|} n^{r-2} \Big) \right).
\end{aligned}$$

Expanding $(n-\deg(h))^{r-1}$ and after some elementary arithmetic manipulations with the error term we obtain that

$$
\begin{aligned}
\sum_{\substack{f \text{ monic} \\ \deg(f)=n}} F(f) &= q^n \sum_{\substack{h \text{ monic} \\ \deg(h)\leq n}} \frac{G(h)}{|h|}\Bigg(\frac{1}{(r-1)!} \\
&\times \left(n^{r-1} - (r-1)n^{r-2}\deg(h) + ... + (-1)^{r-1}\deg(h)^{r-1}\right) + O(n^{r-2})\Bigg) \\
&= \frac{1}{(r-1)!}q^n n^{r-1} \sum_{\substack{h \text{ monic} \\ \deg(h)\leq n}} \frac{G(h)}{|h|} + O\Bigg(q^n n^{r-2} \sum_{\substack{h \text{ monic} \\ \deg(h)\leq n}} \frac{G(h)}{|h|}(\deg(h)+1)\Bigg) \\
&= \frac{1}{(r-1)!}q^n n^{r-1} \sum_{h \text{ monic}} \frac{G(h)}{|h|} + O\Bigg(q^n n^{r-2} \sum_{\substack{h \text{ monic} \\ \deg(h)> n}} \frac{|G(h)|\deg(h)}{|h|}\Bigg) \\
&\quad + O\Bigg(q^n n^{r-2}\Bigg(|G(1)| + \sum_{\substack{h \text{ monic} \\ 1\leq\deg(h)\leq n}} \frac{|G(h)|\deg(h)}{|h|}\Bigg)\Bigg) \\
&= \frac{1}{(r-1)!}G^*(1)q^n n^{r-1} \; + O(G_2^*(1)q^n n^{r-2}),
\end{aligned}
\tag{2.13}
$$

which establishes the desired formula where it is clear the implied constant depends only on r.

The formula given in (2.11) may be proved similarly. Using (2.9) and (2.8) we have that

$$\begin{aligned}
\sum_{\substack{f \text{ monic} \\ \deg(f)=n}} \frac{F(f)}{|f|} &= \sum_{\substack{h \text{ monic} \\ \deg(h)\leq n}} \frac{G(h)}{|h|} + \sum_{\substack{j \text{ monic} \\ \deg(j)=n-\deg(h)}} \frac{d_r(j)}{|j|} \\
&= \sum_{\substack{h \text{ monic} \\ \deg(h)\leq n}} \frac{G(h)}{|h|} \left(\frac{1}{(r-1)!}(n-\deg(h))^{r-1} + O(n^{r-2}) \right) \\
&= \frac{1}{(r-1)!} \sum_{\substack{h \text{ monic} \\ \deg(h)\leq n}} \frac{G(h)}{|h|} \\
&\quad + O\left(n^{r-2} \sum_{\substack{h \text{ monic} \\ \deg(h)\leq n}} \frac{G(h)}{|h|}(\deg(h)+1) \right) \qquad (2.14) \\
&= \frac{1}{(r-1)!} n^{r-1} G^*(1) + O\left(n^{r-2} \sum_{\substack{h \text{ monic} \\ \deg(h)> n}} \frac{|G(h)|\deg(h)}{|h|} \right) \\
&\quad + O\left(n^{r-2} \left(|G(1)| + \sum_{\substack{h \text{ monic} \\ 1\leq\deg(h)\leq n}} \frac{|G(h)|\deg(h)}{|h|} \right) \right) \\
&= \frac{1}{(r-1)!} G^*(1) n^{r-1} + O(G_2^*(1) n^{r-2}).
\end{aligned}$$

Again, the implied constant depends only on r. This concludes the proof of the theorem. □

For our next result we require the following lemma.

Lemma 2.2. *Let $f \in \mathbb{F}_q[t]$, and P denote irreducible monics in $\mathbb{F}_q[t]$. Then we have that*

$$\prod_{P|f} (1 + 1/|P|) = O(\log(\deg(f))), \qquad (2.15)$$

and

$$1 + \sum_{P|f} \frac{\log|P|}{|P|} = O(\log(\deg(f))). \qquad (2.16)$$

Proof. The proof of this lemma follows the same ideas as those based on standard estimates involving primes, for instance, see ([5], Chapter XXII]).

By Mertens' second theorem [7] and its function field analogue proved by Rosen in [9] we have that

$$\sum_{\substack{P \\ \deg P \leq n}} \frac{1}{|P|} = \sum_{k=0}^{n} \left(\frac{1}{q^k} \sum_{\deg P = n} 1 \right) = \sum_{k=0}^{n} \frac{1}{k} \sim \log(\deg f), \tag{2.17}$$

so when dealing with $\sum_{P|f} \frac{1}{|P|}$ we consider irreducible monics dividing f with degree $\leq \log(\deg f)$, then the irreducible monics dividing f with degree $> \log(\deg f)$. The contribution given by irreducible monics of large degree to $\sum_{P|f} \frac{1}{|P|}$ is bounded by a constant and the contribution of monic irreducibles of small degree is bounded by $\log(\log(\deg f))$ by (2.17). It follows that

$$\prod_{P|f} (1 + 1/|P|) = O\left(\exp\left(\sum_{P|f} \frac{1}{|P|} \right) \right) = O\left(\log(\deg f) \right).$$

and so (2.15) is proved. We may prove (2.16) similarly, again through the use of Mertens' Theorem. □

To apply Theorem 2.1 to a function F, it is necessary to determine a relation as in (2.9). In the case of a multiplicative function, it is suitable to use the Dirichlet series. If $F^*(s) = G^*(s)\zeta_A^r(s)$, then it is clear that the arithmetic function G will satisfy (2.9). If $G_2^*(1)$ exists, then the theorem may be used, and G need not be known, since the result requires only $G^*(1)$ and $G_2^*(1)$. This is explained with the following theorem:

Theorem 2.2. *If $r \geq 2$ an integer and N monic, then we have that*

$$\sum_{\substack{f \text{ monic}; \\ \deg(f)=n; \\ (f,N)=1}} \frac{\mu^2(f) d_r(f)}{|f|} = C(N) n^r + O\left(n^{r-1} \log^{r+1}(\deg(N)) \right) \tag{2.18}$$

where the implied constant depends only on r, and

$$C(N) = \prod_{P|N} \left(1 + \frac{r}{|P|} \right)^{-1} \prod_{P} \left(1 - \frac{1}{|P|} \right)^r \left(1 + \frac{r}{|P|} \right).$$

Here the second product runs over all monic irreducibles in $\mathbb{F}_q[t]$.

Proof. Let $\chi_N(f)$ be 1 if $(f, N) = 1$, and 0 otherwise. Then we take $F(f) = \mu^2(f)\chi_N(f)d_r(f)$ and the sum can be written as

$$\sum_{\substack{f \text{ monic} \\ \deg(f)=n}} \frac{F(f)}{|f|}.$$

Now, writing $F^*(s)$ as an Euler product we have that

$$
\begin{aligned}
F^*(s) &= \prod_P \left(1 + \chi_N(P)\frac{r}{|P|^s}\right) = \prod_{P\nmid N}\left(1+\frac{r}{|P|^s}\right) \\
&= \zeta_A^r(s) \prod_P \left(1 - \frac{1}{|P|^s}\right)^r \prod_{P\nmid N}\left(1+\frac{r}{|P|^s}\right) \qquad (2.19)\\
&= \zeta_A^r(s) \prod_P \left(1 - \frac{1}{|P|^s}\right)^r \left(1+\frac{r}{|P|^s}\right) \prod_{P|N}\left(1+\frac{r}{|P|^s}\right)^{-1} \\
&= \zeta_A^r(s) G^*(s).
\end{aligned}
$$

This defines G^* and hence the function G. Note that the form of the Euler product for F^* made it clear what power of the zeta function $\zeta_A(s)$ should be removed from F^*. We now must estimate $G_2^*(1)$. In pursuing this, we rewrite G^* as

$$
\begin{aligned}
G^*(s) &= \prod_{P|N}\left(1-\frac{1}{|P|^s}\right)^r \prod_{P\nmid N}\left(1-\frac{1}{|P|^s}\right)^r \left(1+\frac{r}{|P|^s}\right) \\
&= \prod_{P|N}\left(1-\frac{1}{|P|^s}\right)^r \prod_{P\nmid N}\left(1-\frac{1}{2}r(r+1)\frac{1}{|P|^{2s}}+\ldots\right),
\end{aligned}
$$

so

$$
G_1^*(s) = \prod_{P|N}\left(1+\frac{1}{|P|^s}\right)^r \prod_{P\nmid N}\left(1+\frac{1}{2}r(r+1)\frac{1}{|P|^{2s}}+\ldots\right) = H(s)K(s). \qquad (2.20)
$$

Now,

$$
G_2^*(1) = 1 - G_1^{*'}(1) = 1 - (H'(1)K(1) + H(1)K'(1)) = O(-H'(1) + H(1)),
$$

since $K(1)$ and $K^{'}(1)$ are bounded independently of N.

We have, by the product rule that

$$\begin{aligned}
-H'(s) &= -\frac{d}{ds}\left(\prod_{P|N}(1+\frac{1}{|P|^s})^r\right)\\
&= -\sum_{\substack{i=1;\\ P_1\cdots P_k=N}}^{k}\left(\frac{d}{ds}\left(1+\frac{1}{|P_i|^s}\right)^r\prod_{j\neq i}\left(1+\frac{1}{|P_j|^s}\right)^r\right)\\
&= -\left(\prod_{P|N}\left(1+\frac{1}{|P|^s}\right)^r\right)\left(\sum_{P|N}\left(\frac{d}{ds}\left(1+\frac{1}{|P|^s}\right)^r\Big/\left(1+\frac{1}{|P|^s}\right)^r\right)\right)\\
&= -H(s)\sum_{P|N}\left(-r\frac{1}{|P|^s}\log|P|\left(1+\frac{1}{|P|^s}\right)^{r-1}\Big/\left(1+\frac{1}{|P|^s}\right)^r\right)\\
&= rH(s)\sum_{P|N}\left(\frac{1}{|P|^s}\log|P|\Big/\left(1+\frac{1}{|P|^s}\right)\right).
\end{aligned}$$

Therefore,

$$-H'(1)+H(1)=O\left(\prod_{P|N}\left(1+\frac{1}{|P|}\right)^r\left(1+\sum_{P|N}\frac{\log|P|}{|P|}\right)\right).$$

Using Lemma 2.2, we deduce $\prod_{P|N}(1+1/|P|)=O(\log(\deg(N)))$ and

$$1+\sum_{P|N}\frac{\log|P|}{|P|}=O(\log(\deg(N))).$$

Therefore,

$$G_2^*(1)=O(\log^{r+1}(\deg(N))).$$

Using the fact that (2.19) implies a relation like (2.9), and observing that $G^*(1)=C(N)$, (2.11) yields

$$\sum_{\substack{f\ \text{monic}\\ \deg(f)=n}}\frac{F(f)}{|f|}=C(N)n^r+O\left(n^{r-1}\log^{r+1}(\deg(N))\right),$$

as required. □

In the classical case, by using Selberg's sieve and Theorem 1.1 it is fairly simple to give an upper bound for the number of $n\le M$ for which $n+$

$t_1, \cdots, n + t_r$ are all primes. This bound in turn is the starting point for an elementary solution of the Waring-Goldbach problem. For details see [4]. We are led to believe that the same arguments from [4] with the Selberg's sieve for polynomials over finite fields [12] can be adapted to provide an elementary solution to the function field version of the Waring-Goldbach problem, but we have not pursued this here since this is not the main aim of this note.

3. Extending Burr's Results

In this section we extend Theorem 2.1 to include secondary main terms by improving on the original error term. The two main ingredients that enable us to obtain better error terms and extract secondary terms is the use of the analogue of Rankin's trick in the context of $\mathbb{F}_q[t]$ and the fact that we have exact formulas for the mean values of divisor functions $d_r(f)$ in $\mathbb{F}_q[t]$. These two ingredients with a careful analysis of the main term and the error term lead us to the desired results which are presented below. We present the full details for the cases when f can be expressed as a product of 2 and 3 monic polynomials, i.e., when $r = 2$ and $r = 3$ and then we present the formula for the general case.

Before we establish the formulas with the secondary main terms we need first to discuss the so-called "Rankin's method", which will be used to obtain the desired formulas. In 1938, Rankin [8] introduced an elementary method to bound sums of arithmetic functions by its Dirichlet series. The main idea is that for a multiplicative function f, which takes non-negative values and $\alpha > 0$ we have that

$$\begin{aligned}\sum_{n \leq x} f(n) &\leq \sum_{n \leq x} f(n) \left(\frac{x}{n}\right)^{\alpha} \\ &\leq x^{\alpha} \sum_{n=1}^{\infty} \frac{f(n)}{n^{\alpha}} \\ &= x^{\alpha} \prod_{P} \left(1 + \frac{f(P)}{P^{\alpha}} + \frac{f(P^2)}{P^{2\alpha}} + \cdots\right).\end{aligned} \tag{3.1}$$

It is worth noting that Rankin's method plays a prominent role in several places in analytic number theory, especially in sieve methods (see [6] for more details).

For our purposes, we need an analogue of the above inequality in $\mathbb{F}_q[t]$. It is easy to check that if F is a multiplicative function in $\mathbb{F}_q[t]$, which takes non-negative values, then for any $\alpha > 0$ we have that

$$\begin{aligned}\sum_{\substack{f \text{ monic}\\ \deg(f)\leq x}} F(f) &\leq \sum_{\substack{f \text{ monic}\\ \deg(f)\leq x}} F(f)\left(\frac{q^x}{q^{\deg(f)}}\right)^{\alpha}\\ &\leq q^{\alpha x}\sum_{f \text{ monic}} \frac{F(f)}{|f|^{\alpha}}\\ &= q^{\alpha x}\prod_{P}\left(1+\frac{F(P)}{|P|^{\alpha}}+\frac{F(P^2)}{|P|^{2\alpha}}+\cdots\right).\end{aligned} \tag{3.2}$$

3.1. **The asymptotic formula when $r = 2$.** Let F be a multiplicative function defined as in (2.9), and $f \in \mathbb{F}_q[t]$ be a monic polynomial of degree n. Then from Lemma 2.1 we have that

$$\begin{aligned}\sum_{\deg(f)=n} d_2(f) &= \binom{n+1}{1}q^n\\ &= (n+1)q^n.\end{aligned} \tag{3.3}$$

Therefore, the sum of $F(f)$ over all monic polynomials of degree n is

$$\begin{aligned}\sum_{\substack{f \text{ monic}\\ \deg(f)=n}} F(f) &= \sum_{\substack{h,j \text{ monic}\\ \deg(hj)=n}} G(h)d_2(j)\\ &= \sum_{\substack{h \text{ monic}\\ \deg(h)\leq n}} G(h) \sum_{\substack{g \text{ monic}\\ \deg(j)=n-\deg(h)}} d_2(j).\end{aligned} \tag{3.4}$$

Using (3.3) we have that,

$$\begin{aligned}\sum_{\substack{f \text{ monic}\\ \deg(f)=n}} F(f) &= \sum_{\substack{h \text{ monic}\\ \deg(h)\leq n}} G(h)\Big((n-\deg(h))+1\Big)q^{n-\deg(h)}\\ &= q^n \sum_{\substack{h \text{ monic}\\ \deg(h)\leq n}} \frac{G(h)}{|h|}\Big((n+1)-\ \deg(h)\Big)\\ &= q^n\left(\Big(n+1\Big)\sum_{\substack{h \text{ monic}\\ \deg(h)\leq n}} \frac{G(h)}{|h|} - \sum_{\substack{h \text{ monic}\\ \deg(h)\leq n}} \frac{G(h)}{|h|}\deg(h)\right).\\ &= q^n\, C_{F_2} + O\Big(E_2(n)\Big),\end{aligned} \tag{3.5}$$

where

$$
\begin{aligned}
C_{F_2} &= \big(n+1\big) \sum_{h \text{ monic}} \frac{G(h)}{|h|} - \sum_{h \text{ monic}} \frac{G(h)}{|h|} \deg(h) \\
&= \big(n+1\big) G^*(1) + G^{*\prime}(1),
\end{aligned} \tag{3.6}
$$

and

$$
\begin{aligned}
E_2(n) &= q^n \left(\big(n+1\big) \sum_{\substack{h \text{ monic} \\ \deg(h)>n}} \frac{|G(h)|}{|h|} + \sum_{\substack{h \text{ monic} \\ \deg(h)>n}} \frac{|G(h)|}{|h|} \deg(h) \right) \\
&\ll q^n \sum_{\substack{h \text{ monic} \\ \deg(h)>n}} \frac{|G(h)|}{|h|} \deg(h) \\
&\ll q^{\alpha n} \sum_{h \text{ monic}} \frac{|G(h)|}{|h|^\alpha} \deg(h) \\
&\ll q^{\alpha n} G_2^*(\alpha).
\end{aligned} \tag{3.7}
$$

Note that we have used (3.2) and the fact that $\deg(h) > n$ to obtain the upper bound for $E_2(n)$. Hence

$$
\sum_{\substack{f \text{ monic} \\ \deg(f)=n}} F(f) = q^n \,\big(n+1\big) G^*(1) + q^n G^{*\prime}(1) + O\bigg(q^{n\alpha}\, G_2^*(\alpha) \bigg), \tag{3.8}
$$

where $\alpha \leq 1$. Observe that the formula above includes secondary main terms and $\alpha \leq 1$ so we have obtained a result that improves, in the function field case, Burr's original number field result.

3.2. **The asymptotic formula when $r = 3$ and for a generic r.** Let F, f be defined as before, then from Lemma 2.1 we have that

$$
\begin{aligned}
\sum_{\deg(f)=n} d_3(f) &= \binom{n+2}{2} q^n \\
&= \frac{n^2+3n+2}{2} q^n
\end{aligned} \tag{3.9}
$$

As in previous section, we can write the sum of $F(f)$ over all monic polynomials of degree n as

$$\begin{aligned}\sum_{\substack{f \text{ monic} \\ \deg(f)=n}} F(f) &= q^n \sum_{\substack{h \text{ monic} \\ \deg(h)\leq n}} \frac{G(h)}{|h|}\left(\frac{(n-\deg(h))^2+3(n-\deg(h))+2}{2}\right) \\ &= q^n \left(\Big(\frac{n^2+3n+2}{2}\Big) \sum_{\substack{h \text{ monic} \\ \deg(h)\leq n}} \frac{G(h)}{|h|}\right. \\ &\left. -\Big(\frac{2n+3}{2}\Big) \sum_{\substack{h \text{ monic} \\ \deg(h)\leq n}} \frac{G(h)}{|h|}\deg(h) + \frac{1}{2} \sum_{\substack{h \text{ monic} \\ \deg(h)\leq n}} \frac{G(h)}{|h|}\big(\deg(h)\big)^2\right) \\ &= q^n\, C_{F_3} + O\Big(E_3(n)\Big),\end{aligned} \tag{3.10}$$

where

$$\begin{aligned}C_{F_3} &= \Big(\frac{n^2+3n+2}{2}\Big) \sum_{h \text{ monic}} \frac{G(h)}{|h|} - \Big(\frac{2n+3}{2}\Big) \sum_{h \text{ monic}} \frac{G(h)}{|h|}\deg(h) \\ &+ \frac{1}{2}\sum_{h \text{ monic}} \frac{G(h)}{|h|}\big(\deg(h)\big)^2 \\ &= \Big(\frac{n^2+3n+2}{2}\Big)G^*(1) - \Big(\frac{2n+3}{2}\Big)G^{*\prime}(1) + \frac{1}{2}G^{*\prime\prime}(1),\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}E_3(n) &= q^n\left(\Big(\frac{n^2+3n+2}{2}\Big) \sum_{\substack{h \text{ monic} \\ \deg(h)> n}} \frac{|G(h)|}{|h|} + \Big(\frac{2n+3}{2}\Big) \sum_{\substack{h \text{ monic} \\ \deg(h)> n}} \frac{|G(h)|}{|h|}\deg(h)\right. \\ &\left. + \frac{1}{2}\sum_{h \text{ monic}} \frac{|G(h)|}{|h|^\alpha}\big(\deg(h)\big)^2\right) \\ &\ll q^{\alpha n} \sum_{h \text{ monic}} \frac{|G(h)|}{|h|^\alpha}\Big(\deg(h)\Big)^2 \\ &\ll q^{\alpha n} G_2^{*\prime}(\alpha).\end{aligned} \tag{3.12}$$

Thus,

$$\begin{aligned}\sum_{\substack{f \text{ monic}\\ \deg(f)=n}} F(f) = q^n \Bigg(&\Big(\frac{n^2+3n+2}{2}\Big) G^*(1) + \Big(\frac{2n+3}{2}\Big) G^{*\prime}(1) \\ &+ \frac{1}{2} G^{*\prime\prime}(1) \Bigg) + O\Bigg(q^{\alpha n} G_2^{*\prime}(\alpha) \Bigg),\end{aligned} \tag{3.13}$$

where $\alpha \leq 1$.

It should be observed that the results above can be extended to all divisor functions. In fact, with a little extra work we can deduce that for $r = k$ and $\alpha \leq 1$, we have

$$\begin{aligned}\sum_{\substack{f \text{ monic}\\ \deg(f)=n}} F(f) = &q^n \Bigg(R_k(n) G^*(1) + \sum_{m=1}^{k-1} \frac{1}{m!} R_k^{(m)}(n) G^{*(m)}(1) \Bigg) \\ &+ O\Bigg(q^{\alpha n} G_2^{*(k-2)}(\alpha) \Bigg),\end{aligned} \tag{3.14}$$

where

$$R_k(n) = \binom{n+k-1}{k-1}. \tag{3.15}$$

Note that the derivatives of R_k are taken with respect to n. While, the derivatives of G^* and G_2^* are taken with respect to s.

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