TRUNCATED PRODUCT REPRESENTATIONS FOR L-FUNCTIONS IN THE HYPERELLIPTIC ENSEMBLE

J. C. ANDRADE, S. M. GONEK, AND J. P. KEATING

ABSTRACT. We investigate the approximation of quadratic Dirichlet L-functions over function fields by truncations of their Euler products. We first establish representations for such L-functions as products over prime polynomials times products over their zeros. This is the hybrid formula in function fields. We then prove that partial Euler products are good approximations of an L-function away from its zeros, and that, when the length of the product tends to infinity, we recover the original L-function. We also obtain explicit expressions for the arguments of quadratic Dirichlet L-functions over function fields and for the arguments of their partial Euler products. In the second part of the paper we construct, for each quadratic Dirichlet L-function over a function field, an auxiliary function based on the approximate functional equation that equals the L-function on the critical line. We also construct a parametrized family of approximations of these auxiliary functions, prove the Riemann hypothesis holds for them, and that their zeros are related to those of the associated L-function. Finally, we estimate the counting function for the zeros of this family of approximations, show that these zeros cluster near those of the associated L-function, and that, when the parameter is not too large, almost all the zeros of the approximations are simple.

1. INTRODUCTION

Let $\mathbb{F}_q$ be a finite field with q elements, where q is odd, and let $\mathbb{F}_q[x]$ be the polynomial ring over $\mathbb{F}_q$ in the variable x. We denote by $\mathcal{H}_{2g+1,q}$ the set of monic, square-free polynomials $D \in \mathbb{F}_q[x]$ of degree $2g+1$. This is the hyperelliptic ensemble of the title. Associated with each D is a nontrivial quadratic Dirichlet character χ_D, and a quadratic Dirichlet L-function, which is the same as the Artin L-function corresponding to the character χ_D of $\mathbb{F}_q(x)\big(\sqrt{D(x)}\big)$, where $\mathbb{F}_q(x)$ is

Date: July 20, 2017.

2010 *Mathematics Subject Classification.* Primary 11G20, Secondary 14G10, 11M50.

Key words and phrases. hybrid formula, hyperelliptic curve, finite fields, function fields, random matrix theory, zeros of L-functions.

JCA would like to thank the Department of Mathematics at the University of Rochester for hospitality during a visit where this work started. JCA was partially supported by a Research in Pairs - Scheme 4 London Mathematical Society grant. SMG was supported in part by National Science Foundation Grant DMS-1200582. JPK gratefully acknowledges support under EPSRC Programme Grant EP/K034383/1 (LMF: L-Functions and Modular Forms) and a Royal Society Wolfson Research Merit Award. The authors also would like to thank the American Institute of Mathematics (AIM) where this work was completed.

the rational function field over $\mathbb{F}_q$. These functions will be described more fully in the next section, but in order to explain the contents of this paper, we introduce some of the basic notation now. Excellent general references are Rosen [14] and Thakur [15].

If f is a nonzero polynomial in $\mathbb{F}_q[x]$, we define the norm of f to be $|f| = q^{\deg f}$. If $f = 0$, we set $|f| = 0$. A monic irreducible polynomial P is called a *prime polynomial* or simply a *prime*. The L-function corresponding to the quadratic character χ_D is given by the Euler product

$$L(s,\chi_D) = \prod_{P \text{ prime}} (1-\chi_D(P)|P|^{-s})^{-1} \qquad \text{Re } s > 1, \tag{1}$$

where s is a complex variable. Multiplying out, we obtain the Dirichlet series representation

$$L(s,\chi_D) = \sum_{f \text{ monic}} \frac{\chi_D(f)}{|f|^s} \qquad \text{Re } s > 1. \tag{2}$$

It is often convenient to work with the equivalent functions written in terms of the variable $u = q^{-s}$, namely,

$$\mathcal{L}(u,\chi_D) = \prod_{P \text{ prime}} (1-\chi_D(P)u^{\deg P})^{-1} \qquad |u| < 1/q, \tag{3}$$

and

$$\mathcal{L}(u,\chi_D) = \sum_{f \text{ monic}} \chi_D(f)u^{\deg f} \qquad |u| < 1/q. \tag{4}$$

It turns out that $\mathcal{L}(u,\chi_D)$ is actually a polynomial of degree $2g$ (see Rosen [14], Proposition 4.3), and it satisfies a Riemann hypothesis (see Weil [17]), namely, all its zeros lie on the circle $|u| = q^{-\frac{1}{2}}$. It follows that we may write

$$\mathcal{L}(u,\chi_D) = \prod_{j=1}^{2g}(1-\alpha_j u), \tag{5}$$

where the $\alpha_j = q^{\frac{1}{2}}e(-\theta_j)$, $j = 1, 2 \ldots, 2g$, are the reciprocals of the roots $u_j = q^{-\frac{1}{2}}e(\theta_j)$ of $\mathcal{L}(u,\chi_D)$. (Throughout we write $e(x)$ to denote $e^{2\pi i x}$.) In particular, the restriction $|u| < 1/q$ in (4) (but not in (3)) may be deleted.

Now $\mathcal{L}(u,\chi_D)$ satisfies the functional equation

$$\mathcal{L}(u,\chi_D) = (qu^2)^g \mathcal{L}(1/qu,\chi_D) \tag{6}$$

and also possesses an "approximate functional equation"

$$\mathcal{L}(u,\chi_D) = \sum_{\substack{f \text{ monic} \\ \deg f \le g}} \chi_D(f)u^{\deg f} + (qu^2)^g \sum_{\substack{f \text{ monic} \\ \deg f \le g-1}} \chi_D(f)(qu)^{-\deg f}, \tag{7}$$

which, of course, is exact. The name comes from the analogous formulas in the number field setting which *are* approximations. For instance, for the Riemann zeta function, a symmetrized version of the formula is (see Titchmarsh [16])

$$\zeta(s) = \sum_{n \le \sqrt{t/2\pi}} n^{-s} + \chi(s) \sum_{n \le \sqrt{t/2\pi}} n^{s-1} + E(s),$$

where $0 < \mathrm{Re}\ s < 1,\ t \ge 1$, and $E(s)$ is an error term. The importance of this formula in applications is that it consists of two Dirichlet polynomials of length about $\sqrt{t}$, whereas a more direct approximation (see Titchmarsh [16]) would require a Dirichlet polynomial of length t. The factor $\chi(s)$ is from the functional equation $\zeta(s) = \chi(s)\zeta(1-s)$ and is easy to calculate. Similarly, (7) consists of two pieces of length about g as opposed to a polynomial of length $2g$ (recall (5)). This is analogous because, in a sense, large t in the number field case corresponds to q^{2g}.

In [9] and [10] another type of approximation of the Riemann zeta function and Dirichlet L-functions was constructed. It was based on the approximate functional equation, but used truncations of the L-function's Euler product rather than its Dirichlet series. It was shown, for example, that these approximations satisfy a Riemann hypothesis and are very accurate if one stays away from the zeros of the L-function. Moreover, if the Riemann hypothesis holds for the L-function, the zeros of the approximations converge to those of the L-function as the length of the Euler product tends to infinity. This type of approximation has also been considered in the Physics literature; see, for example, [2]

Our goal in this paper is to carry out a similar construction and analysis in the case of quadratic L-functions for the hyperelliptic ensemble over finite function fields. An advantage we have in this setting is that the Riemann hypothesis is known to hold for such L-functions. This means that all our results are unconditional. Moreover, two interesting differences from the zeta and Dirichlet L-function cases studied in [9] and [10] also emerge. The first is that the proof of the hybrid formula in the function field setting is remarkably straightforward, and moreover the result is simple and exact. The second is that some formulas, such as the one for the argument of an L-function in Theorem 4 below, are surprisingly explicit.

The contents of the paper fall into two parts. The first begins in Section 2 where we give some background on quadratic characters and L-functions, and then prove a hybrid formula for $\mathcal{L}(u, \chi_D)$ (Theorem 1). By this we mean a representation of $\mathcal{L}(u, \chi_D)$ as a product over prime polynomials times a product over its zeros. In Section 3 we prove that partial Euler products $P_K(u, \chi_D)$ approximate $\mathcal{L}(u, \chi_D)$ well inside the disk $|u| \le q^{-\frac{1}{2}}$ when u is not close to any zero $u_j,\ j = 1, 2, \ldots, 2g$, of $\mathcal{L}(u, \chi_D)$, and that at every point in the disc except the u_j's, $\lim_{K\to\infty} P_K(u, \chi_D) = \mathcal{L}(u, \chi_D)$. In Section 4 we obtain explicit expressions for $\arg \mathcal{L}(u, \chi_D)$ and $\arg P_K(u, \chi_D)$ and bound their difference. In the following section we reprove a recent estimate for $\arg \mathcal{L}(u,\chi_D)$ of Faifman and Rudnick [7], and show that if K is sufficiently large, this bound holds for $\arg P_K(u,\chi_D)$ as well. We also reprove, in a slightly different way, another result from [7], an estimate for the counting function $N(\theta,\chi_D)$ of the zeros of $\mathcal{L}(u,\chi_D)$ on the arc $q^{-\frac{1}{2}}e(\phi)$, $0 \le \phi \le \theta \le 1$.

The second part of the paper begins with Section 6. We introduce an auxiliary function $\mathcal{F}(u,\chi_D)$ modeled on the approximate functional equation (7) which equals $\mathcal{L}(u,\chi_D)$ on the all important circle $|u| = q^{-\frac{1}{2}}$ and has the same zeros as $\mathcal{L}(u,\chi_D)$ in the complex plane. In Section 7 we construct a model $\mathcal{F}_K(u,\chi_D)$ of $\mathcal{F}(u,\chi_D)$ built from the truncated Euler products $P_K(u,\chi_D)$. We then show that the Riemann hypothesis holds for $\mathcal{F}_K(u,\chi_D)$, that inside $|u| \le q^{-\frac{1}{2}}$, $\mathcal{F}_K(u,\chi_D)$ approximates $\mathcal{F}(u,\chi_D)$ well when u is away from zeros u_j of $\mathcal{F}(u,\chi_D)$ and K is large enough, and that in this disk $\lim_{K\to\infty} \mathcal{F}_K(u,\chi_D) = \mathcal{F}(u,\chi_D)$ if u is not a u_j. Finally, in the eighth section we estimate the counting function $N_K(\theta,\chi_D)$ of the zeros of $\mathcal{F}_K(u,\chi_D)$, show that the zeros of $\mathcal{F}_K(u,\chi_D)$ cluster around the zeros of $\mathcal{F}(u,\chi_D)$ as $K \to \infty$, and show that when K is not too large, almost all the zeros of $\mathcal{F}_K(u,\chi_D)$ are simple.

In this paper, our main interest is when the cardinality q of the ground field $\mathbb{F}_q$ is fixed and the genus g gets large, i.e., deg$D \to \infty$. It would be interesting to know whether an analysis similar to that of the current paper can be carried out with g fixed and $q \to \infty$.

2. Background on L-functions and a hybrid formula for $\mathcal{L}(u,\chi_D)$

For a prime polynomial P and any $f \in \mathbb{F}_q[x]$, the quadratic residue symbol $\left(\frac{f}{P}\right)$ is defined by

$$\left(\frac{f}{P}\right) = \begin{cases} 0, & \text{if } P \mid f, \\ 1, & \text{if } P \nmid f \text{ and } f \text{ is a square modulo } P, \\ -1, & \text{if } P \nmid f \text{ and } f \text{ is a non square modulo } P. \end{cases}$$

If $Q = P_1^{e_1} P_2^{e_2} \dots P_k^{e_k}$ is the prime factorization of a monic polynomial $Q \in \mathbb{F}_q[x]$, the Jacobi symbol is defined as

$$\left(\frac{f}{Q}\right) = \prod_{i=1}^{k} \left(\frac{f}{P_i}\right)^{e_i}.$$

If $c \in \mathbb{F}_q^*$, then

$$\left(\frac{c}{Q}\right) = c^{((q-1)/2)\deg Q}.$$

If A and B in $\in \mathbb{F}_q[x]$ are monic coprime polynomials, the quadratic reciprocity law, proved by E. Artin, says that

$$\left(\frac{A}{B}\right) = \left(\frac{B}{A}\right)(-1)^{((q-1)/2)\deg A \deg B}.$$

This also holds for A, B not coprime as then both sides equal zero.

For $D \in \mathbb{F}_q[x]$ monic and square-free, we define the *quadratic character* χ_D by

$$\chi_D(f) = \left(\frac{D}{f}\right).$$

For each such character there corresponds an L-function (see (1)-(5) above)

$$L(s, \chi_D) = \sum_{f \text{ monic}} \frac{\chi_D(f)}{|f|^s} = \prod_{P \text{ prime}} (1 - \chi_D(P)|P|^{-s})^{-1} \qquad \text{Re } s > 1.$$

For each D in the hyperelliptic ensemble

$$\mathcal{H}_{2g+1,q} = \{\, D \in \mathbb{F}_q[x] \,:\, D \text{ monic and square-free}, \deg\, D = 2g+1 \,\},$$

there is an associated hyperelliptic curve given in affine form by

$$C_D : y^2 = D(x).$$

These curves are nonsingular and of genus g, and the L-function defined above is related to the zeta function of the curve C_D as follows. Recall that if C is a smooth, projective, connected curve of genus g over $\mathbb{F}_q$, its zeta function is defined as

$$Z_C(u) = \exp\left(\sum_{r=1}^{\infty} N_r(C)\frac{u^r}{r}\right),$$

where $N_r(C)$ is the number of points on C with coordinates in $\mathbb{F}_{q^r}$ (including the point at infinity). Weil [17] proved that

$$Z_C(u) = \frac{P_C(u)}{(1-u)(1-qu)},$$

where $P_C(u)$ is a polynomial of degree $2g$, and he proved the Riemann hypothesis for $Z_C(u)$, which states that all the zeros of $P_C(u)$ lie on the circle $|u| = q^{-\frac{1}{2}}$. In the case of our hyperelliptic curves C_D of odd degree, it turns out that the polynomial $P_{C_D}(u)$ is exactly $\mathcal{L}(u, \chi_D)$ (this was first shown by Artin [1]). As was mentioned above, we may therefore write

$$\mathcal{L}(u, \chi_D) = \prod_{j=1}^{2g}(1 - \alpha_j u) \qquad u \in \mathbb{C},$$

where the $\alpha_j = q^{\frac{1}{2}}e(-\theta_j)$, $j = 1, 2 \ldots, 2g$, are the reciprocals of the roots $u_j = q^{-\frac{1}{2}}e(\theta_j)$ of $\mathcal{L}(u, \chi_D)$.

For a monic polynomial f we write $\Lambda(f) = \deg P$ if $f = P^k$ for some prime P and positive integer k, and $\Lambda(f) = 0$ otherwise. The logarithmic derivative of (3) may then be written

$$\frac{\mathcal{L}'}{\mathcal{L}}(u, \chi_D) = \sum_{P \text{ prime}} \frac{(\deg P)\chi_D(P)u^{\deg P-1}}{1 - \chi_D(P)u^{\deg P}}$$
$$= \sum_{n=1}^{\infty} \Bigg(\sum_{\substack{f \text{ monic} \\ \deg\, f=n}} \Lambda(f)\chi_D(f) \Bigg) u^{n-1}.$$

On the other hand, the logarithmic derivative of (5) is

$$\frac{\mathcal{L}'}{\mathcal{L}}(u, \chi_D) = -\sum_{n=1}^{\infty} \left(\sum_{j=1}^{2g} \alpha_j^n \right) u^{n-1}.$$

Equating these two expressions, we find that

$$-\sum_{j=1}^{2g} e(-n\theta_j) = \frac{1}{q^{n/2}} \sum_{\substack{f \text{ monic} \\ \deg f=n}} \chi_D(f)\Lambda(f). \tag{8}$$

Using this fundamental formula, we prove a version of the hybrid formula of Gonek, Hughes, and Keating [11] (see also [4] and [5]) for $\mathcal{L}(u, \chi_D)$.

Theorem 1 (Hybrid formula for $\mathcal{L}(u, \chi_D)$)**.** *Let $K \geq 0$ be an integer and let*

$$P_K(u, \chi_D) = \exp\Bigg(\sum_{k=1}^{K} \sum_{\substack{f \text{ monic} \\ \deg f=k}} \frac{\Lambda(f)\chi_D(f)u^k}{k} \Bigg), \tag{9}$$

where $\Lambda(f) = \deg P$ if $f = P^n$ for some prime polynomial P, and $\Lambda(f) = 0$ otherwise. Also set

$$Z_K(u, \chi_D) = \exp\Bigg(-\sum_{j=1}^{2g} \Bigg(\sum_{k>K} \frac{(\alpha_j u)^k}{k} \Bigg) \Bigg). \tag{10}$$

Then for $|u| \leq q^{-1/2}$,

$$\mathcal{L}(u, \chi_D) = P_K(u, \chi_D)\, Z_K(u, \chi_D). \tag{11}$$

Remark 1. H. Bui and A. Florea [3] have, independently and at the same time as the current authors, proved a slightly different (weighted) version of the hybrid formula. They use it to calculate low moments of the L-function along the lines of Gonek, Hughes and Keating [11].

Remark 2. One can prove similar formulas for other L-functions defined over finite fields such as for all Dirichlet L-functions $L(s, \chi)$ with χ a Dirichlet character modulo $Q \in \mathbb{F}_q[x]$.

Remark 3. We see that $\lim_{u\to u_j} Z_K(u,\chi_D)=0$ within any sector $|\arg(u-u_j)|<\pi/2-\delta$, where $0<\delta<\pi/2$ is fixed. Thus, we may interpret $Z_K(u_j,\chi_D)$, $j=1,2,\dots,2g$, as zero, even though the infinite series defining $Z_K(u,\chi_D)$ does not converge at u_j. The reader should keep this convention in mind throughout.

Remark 4. If $K=0$, the sum defining $P_K(u,\chi_D)$ is empty, so we interpret $P_0(u,\chi_D)$ as being identically 1. Thus, $\mathcal{L}(u,\chi_D)=Z_0(u,\chi_D)$ for $|u|\le q^{-\frac{1}{2}}$. Indeed, for such u we see that

$$\begin{aligned} Z_0(u,\chi_D) &= \exp\left(-\sum_{j=1}^{2g}\left(\sum_{k=1}^{\infty}\frac{(\alpha_j u)^k}{k}\right)\right) \\ &= \exp\left(\sum_{j=1}^{2g}\log(1-\alpha_j u)\right)=\prod_{j=1}^{2g}(1-\alpha_j u). \end{aligned} \tag{12}$$

In the other direction, for $|u|\le q^{-\frac{1}{2}}$ with small arcs around the zeros u_j removed, we see from (10) that $\lim_{K\to\infty} Z_K(u,\chi_D)=1$ uniformly. Hence, on this set

$$\begin{aligned} \lim_{K\to\infty} P_K(u,\chi_D) &= \exp\left(\sum_{k=1}^{\infty}\sum_{\substack{f \text{ monic}\\ \deg f=k}}\frac{\Lambda(f)\chi_D(f)u^k}{k}\right) \\ &= \exp\left(\sum_{P}\sum_{l=1}^{\infty}\frac{\chi_D(P^l)u^{l\deg P}}{l}\right) \\ &= \prod_{P \text{ prime}}(1-\chi_D(P)u^{\deg P})^{-1}. \end{aligned}$$

In other words, at the extremes, $K=0$ and $K=\infty$, we (essentially) recover expressions for $\mathcal{L}(u,\chi_D)$ as a product over zeros and a product over primes, respectively, from the hybrid formula.

Proof. Assume first that $|u|<q^{-\frac{1}{2}}$. Taking logarithms of both sides of (5) and using the Taylor series for $-\log(1-z),\ |z|<1$, we see that

$$\begin{aligned} \log\mathcal{L}(u,\chi_D) &= -\sum_{j=1}^{2g}\left(\sum_{k=1}^{\infty}\frac{(\alpha_j u)^k}{k}\right) = -\sum_{j=1}^{2g}\left(\sum_{k=1}^{K}+\sum_{k=K+1}^{\infty}\right)\frac{(\alpha_j u)^k}{k} \\ &= -\sum_{k=1}^{K}\frac{u^k}{k}\left(\sum_{j=1}^{2g}\alpha_j^k\right)-\sum_{j=1}^{2g}\left(\sum_{k=K+1}^{\infty}\frac{(\alpha_j u)^k}{k}\right) \end{aligned}$$

By (8), the first double sum equals

$$\sum_{k\le K}\frac{u^k}{k}\left(\sum_{\substack{f \text{ monic}\\ \deg f=k}}\Lambda(f)\chi_D(f)\right).$$

The second is simply

$$-\sum_{j=1}^{2g}\left(\sum_{k>K}\frac{(q^{\frac{1}{2}}e(-\theta_j)u)^k}{k}\right).$$

Thus,

$$\log\mathcal{L}(u,\chi_D)=\sum_{k=1}^{K}\left(\sum_{\substack{f\text{ monic}\\ \deg f=k}}\frac{\Lambda(f)\chi_D(f)u^k}{k}\right)-\sum_{j=1}^{2g}\left(\sum_{k>K}\frac{(q^{\frac{1}{2}}e(-\theta_j)u)^k}{k}\right).$$

Exponentiating this, we obtain (11) for $|u|<q^{-\frac{1}{2}}$. It only remains to treat the case $|u|=q^{-\frac{1}{2}}$. On this circle the first term in (13) is a polynomial in u, so is continuous. The second term, with $u=q^{-\frac{1}{2}}e(\theta)$ and $\theta\neq\theta_j$, $j=1,2,\ldots,2g$, equals

$$-\sum_{j=1}^{2g}\left(\sum_{k>K}\frac{e(k(\theta-\theta_j))}{k}\right). \tag{13}$$

By partial summation, the series $\sum_{k>K}e(k\phi)/k$ converges uniformly for $\delta\le\phi\le 1-\delta$, where $0<\delta<\frac{1}{2}$ is fixed. It follows that (13) is continuous on the circle $|u|=q^{-\frac{1}{2}}$ with the points u_j deleted. Thus $\mathcal{L}(u,\chi_D)$ and the function $P_K(u,\chi_D)\,Z_K(u,\chi_D)$ agree and are analytic in $|u|<q^{-\frac{1}{2}}$, and are continuous on the circle $|u|=q^{-\frac{1}{2}}$ minus the points $u_1,\ldots,u_{2g}$. They therefore agree on this set, and by our interpretation of Z_K as zero in the limit as $u\to u_j$, they agree at these points as well. This completes the proof of the theorem. □

3. Approximation of $\mathcal{L}(u,\chi_D)$ by $P_K(u,\chi_D)$

In this section it simplifies some expressions slightly if we use both the notations $\log g$ and $\log_q g$.

Let $u=q^{-\sigma-it}$ with $\sigma>\frac{1}{2}$ and assume that $K\ge 1$. Then

$$|\log Z_K(u,\chi_D)|=\left|\sum_{j=1}^{2g}\sum_{k>K}\frac{(\alpha_j u)^k}{k}\right|\le 2g\sum_{k>K}\frac{q^{(\frac{1}{2}-\sigma)k}}{k}\le\frac{2g\,q^{(\frac{1}{2}-\sigma)K}}{K(1-q^{\frac{1}{2}-\sigma})}.$$

If we write $\sigma=\dfrac{1}{2}+\dfrac{C\log_q g}{K}$, with $C>0$ and possibly depending on g, then this equals

$$\frac{2g^{1-C}}{K(1-g^{-C/K})}.$$

Since $1-e^{-x}\ge x/2$ for $0\le x\le\frac{1}{2}$, if we assume that $K\ge 2C\log g$ we have

$$1-g^{-C/K}\ge\frac{C\log g}{2K}.$$

Hence the above is

$$\leq \frac{4g^{1-C}}{C\log g}.$$

Using this with Theorem 1, we see that if $C \geq 1$, then

$$\mathcal{L}(u,\chi_D) = P_K(u,\chi_D)\Big(1 + O\Big(\frac{1}{Cg^{C-1}\log g}\Big)\Big).$$

Theorem 2. *Let* $\sigma = \frac{1}{2} + \frac{C\log_q g}{K}$ *with* $C \geq 1$ *(recall that* C *might depend on* g*) and* $K \geq 2C\log g$*. Then for* $|u| \leq q^{-\sigma}$ *we have*

$$\mathcal{L}(u,\chi_D) = P_K(u,\chi_D)\Big(1 + O\Big(\frac{1}{Cg^{C-1}\log g}\Big)\Big).$$

We can prove a similar approximation on the circle $|u| = q^{-1/2}$. Write $u = q^{-\frac{1}{2}}e(\theta)$. Then by (10)

$$Z_K(e(\theta)q^{-\frac{1}{2}},\chi_D) = \exp\Bigg(-\sum_{j=1}^{2g}\Bigg(\sum_{k>K}\frac{e(k(\theta-\theta_j))}{k}\Bigg)\Bigg).$$

By partial summation

$$\sum_{j=1}^{2g}\Bigg(\sum_{k>K}\frac{e(k(\theta-\theta_j))}{k}\Bigg) \ll \frac{1}{K+1}\sum_{j=1}^{2g}\frac{1}{|\sin(\pi(\theta-\theta_j))|}.$$

Let $\|x\| = \min_{n\in\mathbf{Z}}|x-n|$. Then if we assume that

$$\min_{1\leq j\leq 2g}\|\theta-\theta_j\| \geq \frac{c}{2g}$$

and use the estimate $|\sin\pi x| \geq 2\|x\|$, we find that the above is

$$\ll \frac{g^2}{cK}\ .$$

Thus we have

Theorem 3. *Let* $u = q^{-\frac{1}{2}}e(\theta)$*. Suppose that* $c > 0$*,*

$$\min_{1\leq j\leq 2g}\|\theta-\theta_j\| \geq \frac{c}{2g},$$

and $K \geq g^2/c$*. Then*

$$\mathcal{L}(u,\chi_D) = P_K(u,\chi_D)\Big(1 + O\Big(\frac{g^2}{cK}\Big)\Big).$$

Observe that as a consequence of Theorems 2 and 3 we have for $|u| \leq q^{-\frac{1}{2}}$, $u \neq u_j$, $j = 1,2,\ldots,2g$, that

$$\lim_{K\to\infty} P_K(u,\chi_D) \to \mathcal{L}(u,\chi_D).$$

This was pointed out in Remark 3 above, but Theorems 2 and 3 also supply rates of convergence.

4. Explicit expressions for $\arg \mathcal{L}(u, \chi_D)$ and $\arg P_K(u, \chi_D)$

A standard way to define the argument of a quadratic Dirichlet L-function $L(\frac{1}{2} + it, \chi_D)$ when t is not the ordinate of a zero, is by continuous variation starting with the value zero at $s = 2$, moving up the line $\sigma = 2$ as far as $2 + it$, and then horizontally over to $s = \frac{1}{2} + it$. This makes sense for our function field L-functions defined in (1) and (2) as well. For the alternate form $\mathcal{L}(u, \chi_D)$ of $L(s, \chi_D)$, however, this corresponds to continuous variation in the negative sense along the circular arc from q^{-2} to a point $q^{-2}e(-\theta)$, and then along the radius $re(-\theta)$ from $r = q^{-2}$ to $r = q^{-\frac{1}{2}}e(-\theta)$. This would be $\arg \mathcal{L}(q^{-\frac{1}{2}}e(-\theta), \chi_D)$, which is the same as $-\arg \mathcal{L}(q^{-\frac{1}{2}}e(\theta), \chi_D)$, since $\mathcal{L}(u, \chi_D)$ is real for real values of u. Thus, denoting the path consisting of the positively oriented circular arc from q^{-2} to $q^{-2}e(\theta)$ followed by the radial segment from $q^{-2}e(\theta)$ to $q^{-\frac{1}{2}}e(\theta)$ by $\Gamma(\theta)$, we define

$$\arg \mathcal{L}(q^{-\frac{1}{2}}e(\theta), \chi_D) = -\triangle_{\Gamma(\theta)} \ \mathcal{L}(u, \chi_D). \tag{14}$$

If $q^{-\frac{1}{2}}e(\theta)$ happens to be a zero of $\mathcal{L}(u, \chi_D)$, we use the convention that

$$\arg \mathcal{L}(q^{-\frac{1}{2}}e(\theta), \chi_D) = \lim_{\epsilon \to 0^+} \arg \mathcal{L}(q^{-\frac{1}{2}}e(\theta + \epsilon), \chi_D). \tag{15}$$

We also define

$$S(\theta, \chi_D) = \frac{1}{\pi} \arg \ \mathcal{L}(q^{-1/2}e(\theta), \chi_D). \tag{16}$$

Similarly we let

$$\arg P_K(q^{-\frac{1}{2}}e(\theta), \chi_D) = -\triangle_{\Gamma(\theta)} \ P_K(u, \chi_D) \tag{17}$$

and

$$S_K(\theta, \chi_D) = \frac{1}{\pi} \arg \ P_K(q^{-1/2}e(\theta), \chi_D).$$

Our next goal is to obtain alternative expressions for these arguments. From (5) and (14) we find that if $\theta \neq \theta_j$ for any $j = 1, 2, \ldots, 2g$, then

$$\begin{aligned} \arg \mathcal{L}(q^{-1/2}e(\theta), \chi_D) &= - \triangle_{\Gamma(\theta)} \arg \prod_{j=1}^{2g} (1 - \alpha_j u) \\ &= - \sum_{j=1}^{2g} \triangle_{\Gamma(\theta)} \arg(1 - e(\theta - \theta_j)), \end{aligned}$$

where, on the last line, we use the value of the argument in $(-\pi/2, \pi/2)$. Elementary geometric reasoning shows that if $\phi \notin \mathbb{Z}$, then $\arg(1 - e(\phi)) = \pi(\{\phi\} - \frac{1}{2})$, where $\{x\}$ denotes the fractional part of the real number x. Thus

$$\begin{aligned}\arg \mathcal{L}(q^{-1/2}e(\theta),\chi_D) =& \pi\sum_{j=1}^{2g}\big((\{-\theta_j\}-\tfrac{1}{2})-(\{\theta-\theta_j\}-\tfrac{1}{2})\big)\\ =&\pi\sum_{j=1}^{2g}\big(\{-\theta_j\}-\{\theta-\theta_j\}\big).\end{aligned}$$

It follows that for $\theta\neq\theta_j$,

$$S(\theta,\chi_D)=\sum_{j=1}^{2g}\big(\{-\theta_j\}-\{\theta-\theta_j\}\big). \tag{18}$$

If θ does equal θ_i for some i, then by (15) and (16),

$$\begin{aligned}S(\theta_i,\chi_D) &= \lim_{\epsilon\to 0^+}\left(\sum_{j=1}^{2g}(\{-\theta_j\}-\{\theta_i+\epsilon-\theta_j\}\right)\\ &=\sum_{\substack{j=1\\ j\neq i}}^{2g}\big(\{-\theta_j\}-\{\theta_i-\theta_j\}\big)+\lim_{\epsilon\to 0^+}\big(\{-\theta_i\}-\{\theta_i+\epsilon-\theta_i\}\big)\\ &=\sum_{j=1}^{2g}\big(\{-\theta_j\}-\{\theta_i-\theta_j\}\big).\end{aligned}$$

Thus (18) holds whether or not θ is a θ_i.

We can express the formula in both cases in a unified way by using the function

$$s(x)=\begin{cases}\{x\}-\frac{1}{2} & \text{if } x\in\mathbb{R}\setminus\mathbb{Z},\\ 0 & \text{if } x\in\mathbb{Z}.\end{cases}$$

We then clearly have for all θ that

$$S(\theta,\chi_D)=\sum_{j=1}^{2g}\big(s(-\theta_j)-s(\theta-\theta_j)\big). \tag{19}$$

Notice that since $0<\theta_j<1$ for all $j=1,2,\dots,2g$ and since $\theta_{g+j}=1-\theta_j$ for $j=1,2,\dots,g$, we have

$$\begin{aligned}\sum_{j=1}^{2g}s(-\theta_j) =& \sum_{j=1}^{2g}\big(\{-\theta_j\}-\tfrac{1}{2}\big)=\sum_{j=1}^{2g}\big((1-\theta_j)-\tfrac{1}{2}\big)\\ =& g-\sum_{j=1}^{2g}\theta_j=g-\sum_{j=1}^{g}\big(\theta_j+(1-\theta_j)\big)=0.\end{aligned}$$

Thus, (19) is equivalent to

$$S(\theta, \chi_D) = -\sum_{j=1}^{2g} s(\theta - \theta_j).$$

Now it is well known (for example, see Montgomery and Vaughan [13], p. 536) that

$$s(x) = -\sum_{k=1}^{K} \frac{\sin(2\pi x k)}{\pi k} + E_K(x), \tag{20}$$

where for $K \geq 1$,

$$|E_K(x)| \leq \min\left(\frac{1}{2}, \frac{1}{\pi(2K+1)|\sin \pi x|}\right) \leq \min\left(\frac{1}{2}, \frac{1}{4\pi K\|x\|}\right).$$

From this bound we see that the series

$$-\sum_{k=1}^{\infty} \frac{\sin(2\pi x k)}{\pi k}$$

converges pointwise to $s(x)$ when $x \notin \mathbb{Z}$. Moreover, the series clearly converges to $s(x)$ when $x \in \mathbb{Z}$ as well, since then every term is zero. We may therefore summarize the above in

Theorem 4. *For $\theta \in \mathbb{R}$,*

$$S(\theta, \chi_D) = \sum_{j=1}^{2g} \big(s(-\theta_j) - s(\theta - \theta_j)\big) = -\sum_{j=1}^{2g} s(\theta - \theta_j)$$

and

$$S(\theta, \chi_D) = \sum_{j=1}^{2g} \sum_{k=1}^{\infty} \frac{\sin(2\pi(\theta - \theta_j)k)}{\pi k}.$$

Note that the second formula in the theorem is what we would obtain formally from the first line of (12) on taking (14) and basic properties of the θ_j's into account.

We can obtain similar expressions for $\arg P_K(q^{-\frac{1}{2}} e(\theta), \chi_D)$. From (9) and (17)

$$\begin{aligned}\arg P_K(q^{-\frac{1}{2}} e(\theta), \chi_D) &= -\triangle_{\Gamma(\theta)}\ P_K(u, \chi_D) \\ &= -\operatorname{Im} \sum_{k=1}^{K} \frac{e(k\theta)}{k} \Bigg(q^{-k/2} \sum_{\substack{f \text{ monic} \\ \deg f = k}} \Lambda(f) \chi_D(f) \Bigg).\end{aligned}$$

Using (8) to replace the expression in parentheses, we find that

$$\arg P_K(q^{-\frac{1}{2}}e(\theta),\chi_D) = \operatorname{Im} \sum_{k=1}^{K} \frac{e(k\theta)}{k}\Big(\sum_{j=1}^{2g} e(-k\theta_j)\Big)$$
$$= \sum_{j=1}^{2g}\Big(\sum_{k=1}^{K} \frac{\sin(2\pi(\theta-\theta_j)k)}{k}\Big).$$

By (20) this equals

$$\pi \sum_{j=1}^{2g}\big(-s(\theta-\theta_j)+E_K(\theta-\theta_j)\big).$$

Hence, we have

Theorem 5. *For* $K \geq 1$

$$S_K(\theta,\chi_D) = \sum_{j=1}^{2g}\big(-s(\theta-\theta_j)+E_K(\theta-\theta_j)\big)$$

and

$$S_K(\theta,\chi_D) = \sum_{j=1}^{2g}\Big(\sum_{k=1}^{K} \frac{\sin(2\pi(\theta-\theta_j)k)}{\pi k}\Big), \tag{21}$$

where

$$|E_K(\theta-\theta_j)| \leq \min\left(\frac{1}{2}, \frac{1}{4\pi K\|\theta-\theta_j\|}\right).$$

From Theorems 4 and 5 we immediately have

Corollary 6. *For* $K \geq 1$

$$|S(\theta,\chi_D) - S_K(\theta,\chi_D)| \leq \sum_{j=1}^{2g} \min\left(\frac{1}{2}, \frac{1}{4\pi K\|\theta-\theta_j\|}\right). \tag{22}$$

5. THE COUNTING FUNCTION FOR THE ZEROS OF $\mathcal{L}(u,\chi_D)$

It is a simple matter to count the number of zeros of $\mathcal{L}(u,\chi_D)$ on an arc of the circle $|u| = q^{-\frac{1}{2}}$. This was done by a slightly different method by Faifman and Rudnick [7] for the hyperelliptic ensemble $\mathcal{H}_{2g+2,q}$ of even degree monic polynomials. We include a proof because it is short.

Let $N(\theta,\chi_D)$ denote the number of zeros of $\mathcal{L}(u,\chi_D)$ on the circular arc $q^{-\frac{1}{2}}e(\phi),\ \ 0 \leq \phi \leq \theta \leq 1$. That is,

$$N(\theta,\chi_D) = \sum_{\theta_j \leq \theta} 1.$$

For the moment we assume that $\theta \neq \theta_j$, $j = 1, 2, \ldots, 2g$. Let $C(\theta)$ be the positively oriented contour consisting of the circular arc $qe(\phi)$ from $\phi = 0$ to $\phi = \theta$, the radial segment $re(\theta)$ from $r = q$ to $r = q^{-2}$, the circular arc $e(\phi)q^{-2}$ from $\phi = \theta$ to $\phi = 0$, and then the real segment from $r = q^{-2}$ to $r = q$. Then

$$N(\theta, \chi_D) = \frac{1}{2\pi} \triangle_C \arg \mathcal{L}(u, \chi_D).$$

The change in argument along the real segment is zero. Along the outer circular arc from q to $qe(\theta)$, and then along the radius from $qe(\theta)$ to $q^{-1/2}e(\theta)$, we use the functional equation

$$\mathcal{L}(u, \chi_D) = (qu^2)^g \mathcal{L}(1/qu, \chi_D)$$

to see that the change in argument equals $4\pi\theta g + \triangle_{\gamma(\theta)} \arg \mathcal{L}(u, \chi_D)$, where $\gamma(\theta)$ is the contour consisting of the circular arc from $u = q^{-2}$ to $u = q^{-2}e(-\theta)$ and then continuing along the radius $re(-\theta)$ from $r = q^{-2}$ to $q^{-\frac{1}{2}}$. But this is just minus the change in argument along $\Gamma(\theta)$ (see just above (14)), which we defined to be $\arg \mathcal{L}(q^{-\frac{1}{2}} e(\theta), \chi_D)$. Thus the change along the outer part of the contour equals $4\pi\theta g + \arg \mathcal{L}(q^{-\frac{1}{2}} e(\theta), \chi_D)$. By (14) the remaining change in argument also equals $\arg \mathcal{L}(q^{-\frac{1}{2}} e(\theta), \chi_D)$. Thus

$$\begin{aligned} N(\theta, \chi_D) =& \frac{1}{2\pi}(4\pi\theta g + 2 \arg \mathcal{L}(q^{-\frac{1}{2}} e(\theta), \chi_D)) \\ =& 2g\theta + \frac{1}{\pi} \arg \mathcal{L}(q^{-\frac{1}{2}} e(\theta), \chi_D)) \\ =& 2g\theta + S(\theta, \chi_D). \end{aligned} \tag{23}$$

If $\theta = \theta_j$ for some j, our convention (15) means that (23) holds in this case as well. Thus we have proved

Theorem 7. *For all $\theta \in [0, 1]$ we have*

$$N(\theta, \chi_D) = 2g\theta + S(\theta, \chi_D). \tag{24}$$

As a check of this formula we perform the following calculation. According to Theorem 4, if $0 \leq \theta \leq 1$,

$$S(\theta, \chi_D) = \sum_{j=1}^{2g} \big(\{-\theta_j\} - \{\theta - \theta_j\}\big).$$

Now $\{-\theta_j\} = 1 - \theta_j$, and

$$\{\theta - \theta_j\} = \begin{cases} \theta - \theta_j & \text{if} \quad 0 < \theta_j \leq \theta, \\ 1 + \theta - \theta_j & \text{if} \quad \theta < \theta_j < 1. \end{cases}$$

Hence the above is

$$\begin{aligned} S(\theta,\chi_D) =& \sum_{0<\theta_j\le\theta}\big((1-\theta_j)-(\theta-\theta_j)\big)+\sum_{\theta<\theta_j<1}\big((1-\theta_j)-(1+\theta-\theta_j)\big)\\ =& \sum_{0<\theta_j\le\theta}(1-\theta)-\sum_{\theta<\theta_j<1}\theta\\ =& N(\theta,\chi_D)-2g\theta, \end{aligned}$$

which agrees with (24).

6. Upper bounds for $S(\theta,\chi_D)$ and $S_K(\theta,\chi_D)$

Faifman [6] (see also Faifman and Rudnick [7], Proposition 5.1) has shown that for the Hyperelliptic ensemble $\mathcal{H}_{2g+2,q}$,

$$S(\theta,\chi_D)\ll\frac{g}{\log_q g}.$$

This is the analogue of the best known bound for the order of $S(t)=(1/\pi)\arg\zeta(\frac12+it)$ on RH, namely,

$$S(t)\ll\frac{\log t}{\log\log t}.$$

It is clear that the methods of [6] and [7] apply to our ensemble $\mathcal{H}_{2g+1,q}$ as well. In this section we first give a proof of this and then show that the same bound holds for $S_K(\theta,\chi_D)$ if K is sufficiently large with respect to g.

We use the following approximation result which we state without proof (see, for example, Montgomery [12]).

Lemma 8. *Let $I=[\alpha,\beta]$ be an arc in $\mathbb{T}$ with length $\beta-\alpha<1$. Then for any positive integer K there are trigonometric polynomials*

$$T^{\pm}(x)=\sum_{k=-K}^{K}a^{\pm}(k)e(kx)$$

such that

(a) $T^{-}(x)\le\chi_I(x)\le T^{+}(x)$ *for all x,*

(b) $\displaystyle\int_0^1 T^{\pm}(x)dx=\beta-\alpha\pm\frac{1}{K+1}$.

Theorem 9. *For $0<\theta<1$,*

$$S(\theta,\chi_D)\ll\frac{g}{\log_q g}. \tag{25}$$

Proof. For $0 < \theta < 1$ we have

$$N(\theta,\chi_D) = \sum_{j=1}^{2g} \chi_{[0,\theta]}(\theta_j) \le \sum_{j=1}^{2g} T^+(\theta_j)$$
$$= 2ga^+(0) + \sum_{\substack{k=-K\\k\neq 0}}^{K} a^+(k)\Big(\sum_{j=1}^{2g} e(k\theta_j)\Big).$$

By (8) and part (b) of the lemma, we thus see that

$$N(\theta,\chi_D) \le 2g\Big(\theta + \frac{1}{K+1}\Big) - 2\sum_{k=1}^{K} \frac{a^+(k)}{q^{k/2}}\Big(\sum_{\deg f = k} \Lambda(f)\chi_D(f)\Big). \tag{26}$$

Recall that if $f \in L^1(\mathbb{T})$, then

$$|\hat{f}(k)| = \Big|\int_0^1 f(x)e(-kx)dx\Big| \le \int_0^1 |f(x)|dx.$$

Since

$$\hat{\chi}_{[0,\theta]}(k) = e(-k\theta/2)\frac{\sin \pi k\theta}{\pi k}$$

for $k \neq 0$, if we take $f = T^+ - \chi_{[0,\theta]}$, then again by (b) of the lemma

$$\Big|e(-k\theta/2)\frac{\sin \pi k\theta}{\pi k} - a^+(k)\Big| \le \int_0^1 |\chi_{[0,\theta]}(x) - T^+(x)|dx \le \frac{1}{K+1}.$$

Thus,

$$|a_k^+| \le \frac{1}{K+1} + \Big|\frac{\sin \pi k\theta}{\pi k}\Big| \ll \min\Big(\|\theta\|, \frac{1}{k}\Big).$$

From (26) it now follows that

$$S(\theta,\chi_D) = N(\theta,\chi_D) - 2g\theta \le \frac{2g}{K} + O\Big(\sum_{k=1}^{K} \frac{\min(\|\theta\|, k^{-1})}{q^{k/2}}\Big(\sum_{\deg f=k} \Lambda(f)\Big)\Big).$$

By the prime polynomial theorem, the sum in parentheses is equal to q^k. Thus the second term on the right is

$$\ll \sum_{k=1}^{K} \frac{q^{k/2}}{k^2} \ll q^{K/2}.$$

Hence

$$S(\theta,\chi_D) \le \frac{2g}{K} + O(q^{K/2}).$$

The same argument using $T^-(x)$ instead of $T^+(x)$ leads to

$$S(\theta,\chi_D) \ge -\frac{2g}{K} + O(q^{K/2}).$$

Thus

$$S(\theta,\chi_D) \ll \frac{g}{K} + q^{K/2}.$$

Taking $K = \log_q g$, we obtain (25). $\square$

In the case of the zeta function we expect much more to be true, and the heuristic arguments in Farmer, Gonek and Hughes [8] that indicate this also suggest that

$$S(\theta, \chi_D) = O(\sqrt{g \log g}) \qquad \text{and} \qquad S(\theta, \chi_D) = \Omega(\sqrt{g \log g}).$$

To accommodate any eventual improvements in the estimate, we state the next result in terms of a general upper bound $\Phi(g)$ for $S(\theta, \chi_D)$.

Theorem 10. *Suppose that*

$$S(\theta, \chi_D) \ll \Phi(g).$$

Then for $K \geq (g \log g)/\Phi(g)$ *we have*

$$S_K(\theta, \chi_D) \ll \Phi(g). \tag{27}$$

In particular,

$$S_K(\theta, \chi_D) \ll \frac{g}{\log_q g}$$

when $K \geq \log_q g \cdot \log g$

Proof. Set $\Delta = \Phi(g)/g$. The right hand side of (22) is

$$\sum_{m=0}^{[2g/\Delta]+1} \sum_{\substack{j \\ m\Delta \leq \|\theta - \theta_j\| < (m+1)\Delta}} \min\left(\frac{1}{2}, \frac{1}{4\pi K \|\theta - \theta_j\|}\right).$$

By (23), $N(\theta, \chi_D) = 2g\theta + O(\Phi(g))$. Hence, for each m the number of terms in the inner sum over j is $2g\Delta + O(\Phi(g)) \ll \Phi(g)$. The $m = 0$ term therefore contributes $\ll \frac{1}{2}\Phi(g)$, and the remaining terms contribute

$$\ll \frac{1}{K} \sum_{m=1}^{[2g/\Delta]+1} \frac{\Phi(g)}{m\Delta} \ll \frac{\Phi(g) \log(2g/\Delta)}{K\Delta} = \frac{g \log g}{K}.$$

Combining these estimates and taking $K \geq g \log g/\Phi(g)$, we obtain (27). The last assertion of the theorem is an immediate consequence of (27) and (25). $\square$

7. Discussion of a function related to $\mathcal{L}(u, \chi_D)$

We now introduce an auxiliary function $\mathcal{F}(u, \chi_D)$ in order to study $\mathcal{L}(u, \chi_D)$. For $u \in \mathbb{C}$ we define

$$\mathcal{F}(u, \chi_D) = \tfrac{1}{2}\big(\mathcal{L}(u, \chi_D) + (qu^2)^g \mathcal{L}(\overline{u}, \chi_D)\big). \tag{28}$$

Note that $\mathcal{F}(u, \chi_D)$ is not holomorphic although it is harmonic.

The reason we introduce $\mathcal{F}(u,\chi_D)$ is that, unlike $\mathcal{L}(u,\chi_D)$ itself, we can model $\mathcal{F}(u,\chi_D)$ on the closed disk $|u| \le q^{-\frac{1}{2}}$ along the lines of the approximate functional equation (7), using the truncated Euler products $P_K(u,\chi_D)$. By Theorems 2 and 3, $P_K(u,\chi_D)$ is a good approximation of $\mathcal{L}(u,\chi_D)$ when K is large provided we are not too close to a zero of $\mathcal{L}(u,\chi_D)$. This is inevitable because $P_K(u,\chi_D)$ can never vanish. Indeed, Theorem 3 indicates that the closer one is to a zero, the larger K must be to retain a good approximation. Thus, the approximation of $\mathcal{L}(u,\chi_D)$ by $P_K(u,\chi_D)$ is least helpful where we most need it—at the zeros.

Fortunately, "knowing" $\mathcal{F}(u,\chi_D)$ is in many ways the same thing as "knowing" $\mathcal{L}(u,\chi_D)$. For example, on the circle $|u| = q^{-\frac{1}{2}}$,

$$\mathcal{F}(u,\chi_D) = \mathcal{L}(u,\chi_D). \tag{29}$$

To see this observe that $1/qu = \overline{u}$ when $|u| = q^{-\frac{1}{2}}$, and that by the functional equation,

$$\mathcal{L}(u,\chi_D) = (qu^2)^g \mathcal{L}(1/qu,\chi_D) = (qu^2)^g \mathcal{L}(\overline{u},\chi_D).$$

Using this in (28), we obtain (29).

As another example consider the size of $\mathcal{L}(u,\chi_D)$. From (28) it is immediate that

$$\sup_{|u|\le q^{-\frac{1}{2}}} |\mathcal{F}(u,\chi_D)| \le \sup_{|u|\le q^{-\frac{1}{2}}} |\mathcal{L}(u,\chi_D)|.$$

In fact, however, the two quantities are equal. For $\mathcal{F}(u,\chi_D)$, being harmonic, must attain its maximum modulus on the disc on the boundary. However, $\mathcal{F}(u,\chi_D) = \mathcal{L}(u,\chi_D)$ on the boundary, so

$$\sup_{|u|\le q^{-\frac{1}{2}}} |\mathcal{F}(u,\chi_D)| = \sup_{|u|\le q^{-\frac{1}{2}}} |\mathcal{L}(u,\chi_D)|.$$

As a final example we prove

Theorem 11. *The functions $\mathcal{F}(u,\chi_D)$ and $\mathcal{L}(u,\chi_D)$ have the same zeros in $\mathbb{C}$. In particular, the Riemann hypothesis holds for $\mathcal{F}(u,\chi_D)$.*

Proof. Since $\mathcal{F}(u,\chi_D) = \mathcal{L}(u,\chi_D)$, on $|u| = q^{-\frac{1}{2}}$, both functions have the same zeros on this circle. Since $\mathcal{L}(u,\chi_D)$ has no zeros anywhere else, to complete the proof we must show that neither does $\mathcal{F}(u,\chi_D)$.

Suppose, on the contrary, that u_0 is a zero of $\mathcal{F}(u,\chi_D)$ with $|u_0| \ne q^{-\frac{1}{2}}$. Since $\mathcal{L}(u,\chi_D)$ has no zeros off the circle $|u| = q^{-\frac{1}{2}}$, u_0 is not a zero of $\mathcal{L}(u,\chi_D)$. We may therefore write

$$0 = \mathcal{F}(u_0,\chi_D) = \tfrac{1}{2}\mathcal{L}(u_0,\chi_D)\left(1 + (qu_0^2)^g \frac{\mathcal{L}(\overline{u}_0,\chi_D)}{\mathcal{L}(u_0,\chi_D)}\right).$$

The only way the term in parentheses on the right can vanish is if

$$\left|(qu_0^2)^g\frac{\mathcal{L}(\overline{u}_0,\chi_D)}{\mathcal{L}(u_0,\chi_D)}\right| = 1.$$

Now $|\mathcal{L}(\overline{u}_0,\chi_D)/\mathcal{L}(u_0,\chi_D)| = 1$, so this implies that $|u_0| = q^{-\frac{1}{2}}$, a contradiction. □

8. A MODEL OF $\mathcal{F}(u,\chi_D)$

Having shown that we can deduce information about $\mathcal{L}(u,\chi_D)$ from information about $\mathcal{F}(u,\chi_D)$, we now model $\mathcal{F}(u,\chi_D)$ using the Euler product truncations $P_K(u,\chi_D)$. We set

$$\mathcal{F}_K(u,\chi_D) = \tfrac{1}{2}\big(P_K(u,\chi_D) + (qu^2)^g P_K(\overline{u},\chi_D)\big),$$

where $P_K(u,\chi_D)$ is defined in (9). Since $P_K(u,\chi_D)$ has no zeros, we see that $\mathcal{F}_K(u,\chi_D) = 0$ if and only if

$$1 + (qu^2)^g\frac{P_K(\overline{u},\chi_D)}{P_K(u,\chi_D)} = 0. \tag{30}$$

Since $|P_K(\overline{u},\chi_D)/P_K(u,\chi_D)| = 1$, this implies that $|u| = q^{-\frac{1}{2}}$. Thus we have proved

Theorem 12 (The Riemann hypothesis for $\mathcal{F}_K(u,\chi_D)$)**.** *All zeros of $\mathcal{F}_K(u,\chi_D)$ lie on $|u| = q^{-\frac{1}{2}}$.*

As $P_K(u,\chi_D)$ approximates $\mathcal{L}(u,\chi_D)$ well in $|u| \le q^{-\frac{1}{2}}$ when K is large and u is not too close to a zero of $\mathcal{L}(u,\chi_D)$, $\mathcal{F}_K(u,\chi_D)$ approximates $\mathcal{F}(u,\chi_D)$.

Theorem 13. *Let $\sigma = \dfrac{1}{2} + \dfrac{C\log_q g}{K}$ with $C \ge 1$ and $K \ge 2C\log g$. Let $|u| \le q^{-\sigma}$. Then*

$$\mathcal{F}(u,\chi_D) = \mathcal{F}_K(u,\chi_D)\Big(1 + O\Big(\frac{1}{Cg^{C-1}\log g}\Big)\Big). \tag{31}$$

On the circle $u = q^{-\frac{1}{2}}e(\theta)$, if $\min_{1\le j\le 2g}\|\theta-\theta_j\| \ge c/2g$ with $c > 0$ and $K \ge g^2/c$, then

$$\mathcal{F}(u,\chi_D) = \mathcal{F}_K(u,\chi_D)\Big(1 + O\Big(\frac{g^2}{cK}\Big)\Big). \tag{32}$$

Proof. By Theorem 2 and the definition of $\mathcal{F}(u,\chi_D)$,

$$\mathcal{F}(u,\chi_D) = \tfrac{1}{2}\big(P_K(u,\chi_D) + (qu^2)^g P_K(\overline{u},\chi_D)\big)\Big(1 + O\Big(\frac{1}{Cg^{C-1}\log g}\Big)\Big).$$

Equation (31) now follows from the definition of $\mathcal{F}_K(u,\chi_D)$. The proof of (32) is the same except that one uses Theorem 3. □

Corollary 14. *For* $|u| \le q^{-\frac{1}{2}}$, $u \ne u_j$, $j = 1, 2, \ldots, 2g$,

$$\lim_{K\to\infty} \mathcal{F}_K(u, \chi_D) \to \mathcal{F}(u, \chi_D).$$

9. THE ZEROS OF $\mathcal{F}_K(u, \chi_D)$

Since $\mathcal{F}_K(u, \chi_D)$ is a good approximation of $\mathcal{F}(u, \chi_D)$, one wonders whether their zeros are close to one another or whether there are other connections between them. We have seen that both functions satisfy the Riemann hypothesis, so that is a good start. As previously, we write $u_j = q^{-\frac{1}{2}} e(\theta_j)$, $j = 1, 2, \ldots, 2g$, for the zeros of $\mathcal{L}(u, \chi_D)$, where $0 \le \theta_1 \le \theta_2 \le \ldots \le \theta_{2g} < 1$. We denote the zeros of $\mathcal{F}_K(u, \chi_D)$ by $v_j = q^{-\frac{1}{2}} e(\phi_j)$, $j = 1, 2, \ldots$, with $0 \le \phi_1 \le \phi_2 \le \ldots < 1$, leaving open for now the question of their number.

By (30), a necessary and sufficient condition for $v_j = q^{-\frac{1}{2}} e(\phi_j)$ to be a zero of $\mathcal{F}_K(u, \chi_D)$ is that

$$1 + (qv_j^2)^g \frac{P_K(\overline{v}_j, \chi_D)}{P_K(v_j, \chi_D)} = 0.$$

This is equivalent to

$$e^{4\pi i g\phi_j + 2i \arg P_K(v_j, \chi_D)} = -1,$$

or

$$2g\phi_j + \frac{1}{\pi} \arg P_K(v_j, \chi_D) = 2g\phi_j + S_K(\phi_j, \chi_D) \equiv \tfrac{1}{2} \pmod 1.$$

Now as ϕ varies from 0 to 1, the graph of the continuous curve

$$f_K(\phi) = 2g\phi + S_K(\phi, \chi_D) \tag{33}$$

traverses a vertical distance greater than or equal to $f_K(1) - f_K(0) = 2g - 0 = 2g$. Thus it intersects at least $2g$ of the horizontal lines $y = k + \frac{1}{2}$, $k \in \mathbb{Z}$, possibly more than once. We let these values be $\phi_1, \phi_2, \ldots$ in increasing order. Then the points $v_j = q^{-1/2} e(\phi_j)$ are the *distinct* zeros of $\mathcal{F}_K(u, \chi_D)$. Thus, $\mathcal{F}(u, \chi_D)$ has $2g$ zeros, counting multiplicities, and $\mathcal{F}_K(u, \chi_D)$ has at least $2g$ distinct zeros. Similarly, we see that the number of zeros of $\mathcal{F}_K(u, \chi_D)$ on any arc $u = q^{-\frac{1}{2}} e(\phi)$, $0 \le \phi \le \theta$, where $0 \le \theta < 1$, is

$$N_K(\theta, \chi_D) \ge 2g\theta + S_K(\theta, \chi_D) + O(1).$$

Combining this with Theorem 10 we obtain

Theorem 15. *Let* $K \ge g \log g / \Phi(g)$. *Then*

$$N_K(\theta, \chi_D) \ge 2g\theta + O(\Phi(g)).$$

Next we show that the zeros of $\mathcal{F}_K(u,\chi_D)$ are close to those of $\mathcal{F}(u,\chi_D)$ when K is large. We saw that $\mathcal{F}_K(u,\chi_D)$ has a zero at $u = q^{-\frac{1}{2}}e(\theta)$ if and only if

$$f_K(\theta) = 2g\theta + S_K(\theta,\chi_D) \equiv \tfrac{1}{2} \pmod 1. \tag{34}$$

Thus

$$\begin{aligned} f_K(\theta) =& 2g\theta + S(\theta,\chi_D) + (S_K(\theta,\chi_D) - S(\theta,\chi_D) \\ =& N(\theta,\chi_D) + (S_K(\theta,\chi_D) - S(\theta,\chi_D)). \end{aligned}$$

Suppose now that θ_i and θ_{i+1} are arguments corresponding to distinct consecutive zeros of $\mathcal{F}(u,\chi_D)$, and let $0 < \Delta < \frac{1}{2}(\theta_{i+1}-\theta_i)$. Then on the interval $I = [\theta_i+\Delta, \theta_{i+1}-\Delta]$, $N(\theta,\chi_D)$ is an integer. Thus, if $|S_K(\theta,\chi_D) - S(\theta,\chi_D)| < \frac{1}{2}$ on I, then (34) cannot hold. By (22), if $\theta \in I$ we have

$$\begin{aligned} |S(\theta,\chi_D) - S_K(\theta,\chi_D)| \leq& \sum_{j=1}^{2g} \min\left(\frac{1}{2}, \frac{1}{4\pi K\|\theta-\theta_j\|}\right) \\ \leq& \frac{g}{2\pi K\Delta}. \end{aligned}$$

Therefore, if $K > g/\pi\Delta$, then $S_K(\theta,\chi_D) - S(\theta,\chi_D) < \frac{1}{2}$ on I. We have thus proved

Theorem 16. *Let θ_i and θ_{i+1} correspond to distinct consecutive zeros of $\mathcal{F}(u,\chi_D)$, and let $0 < \Delta < \frac{1}{2}(\theta_{i+1}-\theta_i)$. Then if $K > g/\pi\Delta$, $\mathcal{F}_K(u,\chi_D)$ has no zero on the interval $I = [\theta_i+\Delta, \theta_{i+1}-\Delta]$. In particular, the zeros of $\mathcal{F}_K(u,\chi_D)$ cluster around the zeros of $\mathcal{F}(u,\chi_D)$ as $K \to \infty$.*

Our last theorem concerns the simplicity of zeros of $\mathcal{F}_K(u,\chi_D)$. We may write

$$\mathcal{F}_K(q^{-\frac{1}{2}}e(\theta),\chi_D) = \tfrac{1}{2}P_K(q^{-\frac{1}{2}}e(\theta),\chi_D)(1+e(f_K(\theta)))$$

with $f_K(\theta)$ defined in (33). Recall that θ corresponds to a zero of $\mathcal{F}_K(u,\chi_D)$ if and only if (34) holds. This zero will be simple if and only if

$$\frac{d}{d\theta}\mathcal{F}_K(q^{-\frac{1}{2}}e(\theta),\chi_D) \neq 0,$$

which is easily seen to be equivalent to

$$\frac{df_K(\theta)}{d\theta} \neq 0.$$

By (33) and (21),

$$\frac{df_K(\theta)}{d\theta} = 2g + 2\sum_{j=1}^{2g}\left(\sum_{k=1}^{K} \cos(2\pi(\theta-\theta_j)k)\right).$$

The right hand side is a trigonometric polynomial in θ of degree K so it has at most $2K$ zeros. Now, by Theorem 15, if $K \geq g\log g/\Phi(g)$, then $\mathcal{F}_K(u,\chi_D)$ has

$\geq 2g(1+o(1))$ zeros. Thus, if we also have $K=o(g)$, then at most $o(g)$ of these will be multiple. Taking $\Phi(g)=g/\log_q g$, we deduce

Theorem 17. *If* $\log g \log_q g \leq K = o(g)$*, then almost all zeros of* $\mathcal{F}_K(u,\chi_D)$ *are simple.*

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF EXETER, NORTH PARK ROAD, EXETER EX4 4QF, UK

E-mail address: `j.c.andrade@exeter.ac.uk`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ROCHESTER, ROCHESTER, NY 14627, USA

E-mail address: gonek@math.rochester.edu

SCHOOL OF MATHEMATICS, UNIVERSITY OF BRISTOL, BRISTOL BS8 1TW, UK

E-mail address: j.p.keating@bristol.ac.uk