# EXERCISES: ANALYTIC NUMBER THEORY IN $\mathbb{F}_q[x]$ EXETER 2018

**Exercise 1.** *The divisor function $d_k(f)$ for a monic polynomial $f \in \mathbb{F}_q[x]$ is the number of $k$-tuples $(a_1, \dots, a_k) \in \mathbb{F}_q[x]^k$ of monic polynomials so that $f = a_1 \cdot \dots \cdot a_k$.*

*Show that for* $\mathrm{Re}(s) > 1$,

$$\sum_{f \text{ monic}} \frac{d_k(f)}{|f|^s} = \zeta_q(s)^k.$$

**Exercise 2.** *The Möbius function for $\mathbb{F}_q[x]$ is defined as $\mu(f) = (-1)^k$ if $f = cP_1 \cdot \dots \cdot P_k$ is a product of $k$* distinct *monic irreducibles, $c \in \mathbb{F}_q^*$, and $\mu(f) = 0$ otherwise. Show that for* $\mathrm{Re}(s) > 1$,

$$\sum_{f \text{ monic}} \frac{\mu(f)}{|f|^s} = \frac{1}{\zeta_q(s)}.$$

**Exercise 3.** *Show that*

$$\sum_{d|f} \Lambda(d) = \deg f$$

**Exercise 4.** *Let $\mathbb{F}_q[x]$ be the polynomial ring over a finite field $\mathbb{F}_q$ of $q$ elements. The divisor function $d_k(f)$ for a monic polynomial $f \in \mathbb{F}_q[x]$ is the number of $k$-tuples $(a_1, \dots, a_k) \in \mathbb{F}_q[x]^k$ of monic polynomials so that $f = a_1 \cdot \dots \cdot a_k$.*

*Show that for $k \geq 2$, the mean value of $d_k(f)$ over all monic polynomials of degree $n$ is given by the binomial coefficient*

$$\frac{1}{q^n} \sum_{\substack{\deg f = n \\ f \text{ monic}}} d_k(f) = \binom{n+k-1}{k-1} = \frac{(n+k-1) \cdot \ldots \cdot (n+1)}{(k-1)!}$$

**Exercise 5.** *Show that*

$$\sum_{\substack{\deg f = n \\ f \text{ monic}}} \mu(f) = 0, \quad n \geq 2$$

**Exercise 6.** *Show that*

$$\sum_{\deg P \leq N} \frac{1}{|P|} \sim \log N, \qquad N \to \infty$$

*the sum over all prime polynomials (monic irreducibles) and in particular that* $\sum_P 1/|P| = \infty$.

**Exercise 7.** *The cycle structure of a permutation* $\sigma$ *of* $n$ *letters is* $\lambda(\sigma) = (\lambda_1, \dots, \lambda_n)$ *if in the decomposition of* $\sigma$ *as a product of disjoint cycles, there are* $\lambda_j$ *cycles of length* $j$. *In particular* $\lambda_1(\sigma)$ *is the number of fixed points of* $\sigma$.

*For each partition* $\lambda \vdash n$, *denote by* $p(\lambda)$ *the probability that a random permutation on* $n$ *letters has cycle structure* $\lambda$:

$$p(\lambda) = \frac{\#\{\sigma \in S_n : \lambda(\sigma) = \lambda\}}{\#S_n} .$$

*Show that*

$$p(\lambda) = \prod_{j=1}^{n} \frac{1}{j^{\lambda_j} \cdot \lambda_j!}$$

*In particular, this shows that the proportion of* $n$*-cycles in the symmetric group* $S_n$ *is* $1/n$.

**Exercise 8.** *For* $f \in \mathbb{F}_q[x]$ *of positive degree* $n$, *we say its cycle structure is* $\lambda(f) = (\lambda_1, \dots, \lambda_n)$ *if in the prime decomposition* $f = \prod_\alpha P_\alpha$ *(we allow repetition), we have* $\#\{\alpha : \deg P_\alpha = j\} = \lambda_j$. *In particular* $\deg f = \sum_j j\lambda_j$. *Thus we get a partition of* $\deg f$, *which we denote by* $\lambda(f)$. *For instance,* $f$ *is prime if and only if* $\lambda(f) = (0, 0, \dots, 0, 1)$.

*Given a partition* $\lambda \vdash n$, *show that the probability that a random monic polynomial* $f$ *of degree* $n$ *has cycle structure* $\lambda$ *is asymptotic, as* $q \to \infty$, *to the probability that a random permutation of* $n$ *letters has that cycle structure:*

$$\frac{1}{q^n} \#\{f \text{ monic}, \deg f = n : \lambda(f) = \lambda\} = p(\lambda)\Big(1 + O(\frac{1}{q})\Big).$$

*Hint: start with primes, where the statement is just the Prime Polynomial Theorem.*