

An INTRODUCTION TO THE
FUNCTION FIELD APPROACH TO
THE CLASSICAL RIEMANN HYPOTHESIS.

LECTURE - I

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CONTENT OF THE TALK.

1. INTRODUCTION AND AIMS.

2. ZETA - FUNCTIONS ATTACHED TO
CURVES.

3. THE EXPLICIT FORMULAE FOR $\zeta_K(s)$.

4. WEIL'S REFORMULATION OF THE RIEMANN
HYPOTHESIS FOR $\zeta_K(s)$.

5. SKETCH OF THE PROOF OF RIEMANN HYPOTHESIS
FOR $\zeta_K(s)$.

6. TRYING TO APPLY WEIL'S PROOF FOR $\zeta(s)$.

1. INTRODUCTION

→ The Riemann zeta function

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \Re(s) > 1.$$

→ Euler product representation in the same region

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$

Some Properties of $\zeta(s)$.

i) $\zeta(s)$ has analytic continuation to \mathbb{C}

↳ simple pole at $s=1$ with residue 1.

ii) Functional Equation:

$$\xi(s) := \frac{s(s-1)}{2} \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

Then $\xi(s) = \xi(1-s)$ or,

$$\zeta(s) = \chi(s) \zeta(1-s), \text{ where}$$

$$\chi(s) = \pi^{s-1/2} \frac{\Gamma\left(\frac{1}{2} - \frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}.$$

Location of the zeros of $\zeta(s)$.

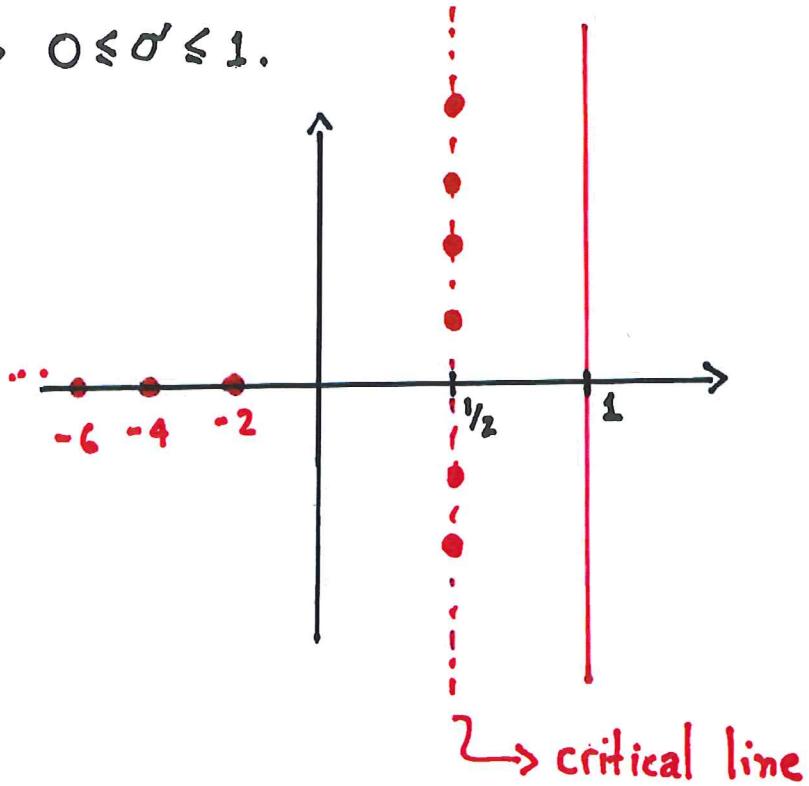
(2)

i) The trivial zeros,

$$\zeta(-2k) = 0 \quad \text{for all integers } k \geq 1.$$

ii) The non-trivial zeros. ($s = \sigma + it$)

→ are the complex zeros $\rho = \beta + it$ which lie on the critical strip $0 \leq \sigma \leq 1$.



Riemann Hypothesis (R.H.): The nontrivial zeros of $\zeta(s)$ have real part equal to $1/2$.

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Favorable Evidences to R.H.

- i) the first 10 billion zeros are on the line (van de Lune)
- ii) millions of zeros are on the line near the numbers $10^{20}, 10^{21}, 10^{22}$ (Odlyzko)
- iii) Proportion of Zeros:
 - * $\frac{1}{3}$ of the zeros are on the line. (Levinson)
 - * more than 40% " " " " (B. Conrey)
 - * more than 41% " " " " (Boi, Conrey, Young)
- iv) symmetry or order in the primes :

$$\text{R.H.} \iff \pi(x) = \frac{x}{\log x} + O(x^{\frac{1}{2}+\epsilon})$$

Some Approaches to the R.H.

i) Hilbert - Pólya Conjecture

→ Find a Hermitian operator whose eigenvalues are the non trivial zeros of $\zeta(s)$. Then the

$$\zeta(\frac{1}{2}+it)$$

\Rightarrow R.H. true (Hermitian operators have real eigenvalues)

ii) Pólya Analysis.

$\Xi(t) := \xi(\frac{1}{2} + it) \Rightarrow \text{R.H. : all zeros of } \Xi(t) \text{ are real.}$

$$\Phi(t) := \int_{-\infty}^{\infty} \Xi(u) e^{itu} du = \sum_{n=1}^{\infty} (2n^4 \pi^2 \exp(\frac{9t}{2}) - 3n^2 \pi \exp(\frac{5t}{2})) \\ \times (\exp(-\pi n^2 e^{2t}))$$

→ Fourier transform. $\Phi(t)$ and $\Phi'(t) > 0$
for $t > 0$.

Idea: study classes of functions whose Fourier transform have all real zeros and then prove $\Xi(t)$ is in this class.

iii) Explicit Formula

Suppose h even function, holomorphic in $|Y(t)| \leq \frac{1}{2} + \delta$ and satisfies $h(t) = O((1+|t|)^{-2-\delta})$ for some $\delta > 0$, let

$$g(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(r) e^{-iur} dr$$

and $p = \frac{1}{2} + ir$, $r \in \mathbb{C}$. be such that $\zeta(p) = 0$. Then

$$\sum_p h(p) = 2h(\frac{1}{2}) - g(0) \log \pi + \frac{1}{2\pi} \int_{-\infty}^{\infty} h(r) \frac{T'}{\pi} (\frac{1}{4} + \frac{1}{2}ir) dr \\ - 2 \sum_{n=1}^{\infty} \frac{\Delta(n)}{\sqrt{n}} g(\log n).$$

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R.H. says that ρ is real for all ρ .

(Weil)

$$\text{R.H. holds} \iff \sum_p h(p) > 0$$

for every admissible function
 h of the form $h(r) = h_0(r)\overline{h_0(\bar{r})}$

iv) Positivity Criterion.

$$\text{R.H.} \iff \lambda_n \geq 0$$

for $n = 1, 2, \dots$ where

$$\lambda_n = \frac{1}{(n-1)!} \left. \frac{d^n}{ds^n} (s^{n-1} \log \xi(s)) \right|_{s=1}$$

$$\lambda_n = \sum_p \left(1 - \left(1 - \frac{1}{p} \right)^n \right) \quad (\text{Li})$$

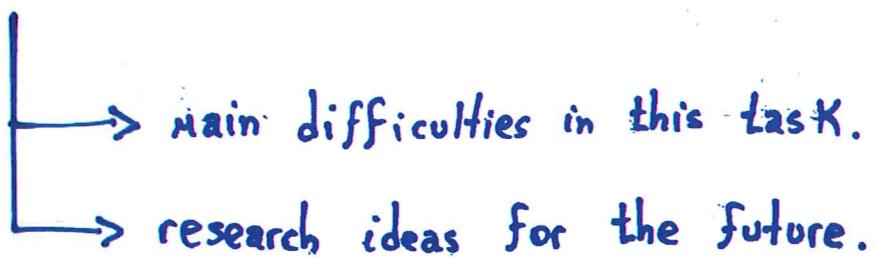
v) Riemann Hypothesis for Curves.

The Riemann Hypothesis is valid for zeta-functions attached to curves over finite fields.

→ the most convincing reason.

Aim of this talk:

- Proof of R.H. Curves. (Main Ideas)
- How far we can mimic the proof given above for $S(s)$.



2. ZETA-FUNCTIONS ATTACHED TO CURVES.

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Historical Background

- Initiated by Artin in his thesis (1921)
 - analytic properties
 - Riemann Hypothesis ??
- Schmidt established some analytic properties. (1931)
- Hasse proved the R.H. for elliptic curves. (1933)
 - ideas from Algebraic Geometry.
- Weil proved the R.H. for general curves (1942)
- Simpler proofs of R.H. by Bombieri (1974) and Voloch (1984).
- Deligne established the R.H. for general varieties (1974).
- Recent proof of R.H. for curves by N. Katz (2013/2014)

Zeta-function of Curves over Finite Fields.

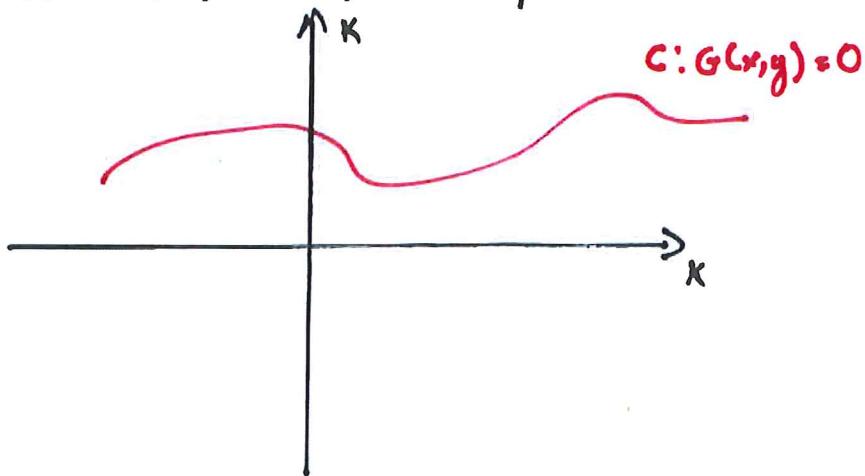
Let $q = p^f$, p prime. Denote by \mathbb{F}_q - finite field with q - elements.

- Consider a finitely generated extension K of \mathbb{F}_q of transcendence degree one.
- Assume that \mathbb{F}_q is the largest finite field contained in K .

Theorem (Primitive Element): There are $x, y \in K$ so that $K = \mathbb{F}_q(x, y)$, x is transcendental over \mathbb{F}_q and there is an irreducible polynomial $G \in \mathbb{F}_q[x, y]$ so that $G(x, y) = 0$.

$\Rightarrow K$ is the algebraic extension of $\mathbb{F}_q(x)$ generated by y satisfying $G(x, y) = 0$. considered as polynomial in one variable over $\mathbb{F}_q(x)$.

- $G(x, y) = 0$ defines a curve C in the plane, so we can consider K as the field of meromorphic functions on this curve.



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e.g.: $F = \mathbb{C}$ these fields are in analysis the fields of meromorphic function on compact Riemannian surfaces.

- notion of a place of a curve.

\hookrightarrow refinement of the idea of a point on a curve.

Let $A = \mathbb{F}_q[x]$ and $\mathbb{F}_q((x))$

$$\sum (\mathbb{F}_q(x)) = \{ \text{FEA, monic and irreducible} \} \cup \{\infty\}.$$

$\mathbb{F}_q^d((t)) =$ the field of power series over \mathbb{F}_q^d .

$$\sum_{k=0}^{\infty} a_k t^k, a_k \in \mathbb{F}_q^d$$

- Let g be any monic irreducible polynomial, s.t. $\deg(g) = d$.

Associated to g there is an embedding of $\mathbb{F}_q(x) \hookrightarrow \mathbb{F}_q^d((t))$

$$\sigma_g : \mathbb{F}_q(x) \hookrightarrow \mathbb{F}_q^d((t))$$

→ this embedding is done in the following way :

- i) we embed the subfield $\mathbb{F}_q(g(x)) \hookrightarrow \mathbb{F}_q^d((t))$

$$\sigma_g(g(x)) = t \quad \text{sending the generator } g(x) \text{ to } t$$

⇒ We obtain an embedding $\mathbb{F}_q(x) \hookrightarrow \mathbb{F}_q(x) \otimes_{\mathbb{F}_q(g(x))} \mathbb{F}_q^d((t))$

Remark: $\mathbb{F}_q((x)) \otimes_{\mathbb{F}_q((x))} \mathbb{F}_q((t))$ is the ring that corresponds to considering the extension of $\mathbb{F}_q((t))$ given by the algebraic equation $g(x) - t$.

- there exists a power series $x(t)$ with coefficients in the splitting field \mathbb{F}_{q^d} of g over \mathbb{F}_q giving a solution.
 - Newton's method.
 - \mathbb{F}_q perfect field.

$$\Rightarrow \mathbb{F}_q((x)) \otimes_{\mathbb{F}_q((x))} \mathbb{F}_q((t)) \simeq \mathbb{F}_{q^d}((t))$$

Some Useful Definitions.

Def.: $\text{ord}: \mathbb{F}_{q^d}((t))^{\times} \rightarrow \mathbb{Z}$

$$\text{ord}(\sum a_n t^n) = \min \{n \in \mathbb{Z} : a_n \neq 0\}$$

Def.: $\| \cdot \|: \mathbb{F}_{q^d}((t)) \longrightarrow \mathbb{R}_+$

$$\|0\| = 0$$

$$\|f\| = q^{1-\text{ord}(f)} \quad f \neq 0.$$

Proposition: (i) $\|x\| = 0 \iff x = 0$

(ii) $\|x+y\| \leq \min(\|x\|, \|y\|)$

(iii) $\|xy\| = \|x\| \|y\|$

(iv) If $d: \mathbb{F}_{q^1}((t)) \times \mathbb{F}_{q^1}((t)) \rightarrow \mathbb{R}$

$d(x, y) = \|x-y\|$. Then

$(\mathbb{F}_{q^1}((t)), d)$ is a complete metric space

Returning to the Embedding Story.

- ~~$\sigma_g[\mathbb{F}_q(x)]$~~ is dense in $\mathbb{F}_{q^1}((t))$
 ↳ image

(ii) ~~$\sigma_\infty(\infty)$~~ To the ∞ symbol we ascribe the embedding

$$\mathbb{F}_q(x) \hookrightarrow \mathbb{F}_{q^1}((t))$$

$$\sigma_\infty(x) = t^{-1}$$

Theorem: Let $\sigma: \mathbb{F}_q(x) \hookrightarrow \mathbb{F}_{q^1}((t))$ be any embedding s.t.

$\sigma[\mathbb{F}_q(x)]$ has dense image. Then σ is of the form

(i) or (ii) composed with an automorphism of $\mathbb{F}_{q^1}((t))$

Corollary: The automorphism above preserves the norm.

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Places of an arbitrary field $K = \mathbb{F}_q((x,y))$

Def.: A place w of K is an equivalence class of embeddings

$K \hookrightarrow \mathbb{F}_q'((t))$ with dense image. Where two embeddings are to be considered equivalent if they differ by an automorphism of $\mathbb{F}_q'((t))$ over \mathbb{F}_q .

→ embeddings \mathbb{F}_q -maps.

- $K = \mathbb{F}_q(x, y)$
- consider a place w of $\mathbb{F}_q(x)$ (dense image)
- ⇒ this is represented by an embedding $\mathbb{F}_q(x) \hookrightarrow \mathbb{F}_q'((t))$
- ⇒ we obtain an embedding $K \hookrightarrow K \otimes_{\mathbb{F}_q(x)} \mathbb{F}_q'((t))$

Proposition: (i) $K \otimes_{\mathbb{F}_q(x)} \mathbb{F}_q'((t)) \simeq \bigoplus_i \mathbb{F}_q^{(i)}((t'))$

(ii) The projection $K \otimes_{\mathbb{F}_q(x)} \mathbb{F}_q'((t))$ into any of $\mathbb{F}_q^{(i)}((t'))$ yields an embedding as above and any embedding is equivalent to one of these projections.

Set of places of $K := \sum(w)$

- Suppose $w \in \Sigma(\kappa)$ and that w is represented by an embedding of $K \hookrightarrow \mathbb{F}_{q'}((t))$.

$$\Rightarrow q' = q^{\mathcal{F}(w)}, \text{ where}$$

$$\mathcal{F}: \Sigma(\kappa) \rightarrow \mathbb{N} \quad (\text{degree of } w)$$

Relationship Between $\Sigma(\kappa)$ and points on C .

- $C: G(x,y) = 0 \rightarrow$ in general not smooth.

Theorem: There is a smooth curve C' in some projective space with function field K defined over $\mathbb{F}_{q'}$.

Let $C = C'$. (abuse of language)

- Let P be a point of C defined over $\mathbb{F}_{q'}$, i.e., a map of $\text{Spec}(\mathbb{F}_{q'})$ into C .

$$\Rightarrow P: \text{Spec}(\mathbb{F}_{q'}) \rightarrow C$$

* as C is smooth this can be extended to a map of a formal neighbourhood of P into C , i.e., $\text{Spec}(\mathbb{F}_{q'}[[t]])$ into the formal completion of C .

\iff (fractions) this is an embedding $K \hookrightarrow \mathbb{F}_{q'}((t))$

- P and P' on C give the same embedding if they are conjugate under the Galois group of $\mathbb{F}_q^{\times}/\mathbb{F}_q$

$\Leftrightarrow w \in \Sigma(K)$ corresponds to an irreducible divisor on C which splits into $f(w)$ points over $\mathbb{F}_q^{f(w)}$.

Zeta function of Function Fields.

Let $K = \mathbb{F}_q(x, y)$ be a given function field.

Def.: The zeta-function of K is defined to be

$$\zeta_K(s) = \prod_{w \in \Sigma(K)} (1 - q^{-f(w)s})^{-1}, \quad \Re(s) > 1.$$

In the classical case,

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}.$$

→ places of \mathbb{Q} corresponds to embeddings $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$ as p ranges over the prime numbers. + $\mathbb{Q} \hookrightarrow \mathbb{R}$ for the prime at infinity which gives the $\pi^{-s/2} T(s/2)$

Theorem : Suppose that the genus of the curve C is g .

Then :

(i) Rationality : (analytic continuation to \mathbb{C})

$$\boxed{\zeta_K(s) = \frac{P_c(q^{-s})}{(1-q^{-s})(1-q^{1-s})}, \quad u = q^{-s}}$$

where $P_c \in \mathbb{Z}[u]$ is a polynomial of degree $2g$ with $P_c(0) = 1$.

Proved by Artin and Dwork - [Rosen Book]

(ii) Functional Equation : Let $\xi_K(s) = q^{(g-1)s} \zeta_K(s)$. Then (Weil)

$$\xi_K(s) = \xi_K(1-s), \quad \text{i.e.,}$$

$$\boxed{\zeta_K(1-s) = q^{(1-g)(1-2s)} \zeta_K(s).}$$

Riemann-Roch Theorem / Poincaré Duality. [Rosen-Book].

- By, there is a set A such that

$$\boxed{P_c(u) = \prod_{\alpha \in A} (1-\alpha u) = \prod_{i=1}^{2g} (1-\pi_i u)},$$

π_i are the inverse roots of $P_c(u)$.

→ Functional Equation for $P_c(u)$:

$$P_c(u) = (q u^2)^g P_c\left(\frac{1}{q u}\right)$$

(iii) Riemann Hypothesis (holds - Weil)

All the roots of $\zeta_K(s)$ lie on the line $\Re(s) = \frac{1}{2}$.

Equivalently, the inverse roots of $P_c(u)$, i.e., $\alpha \in A$
then $|\alpha| = q^{\frac{1}{2}}$.

Reference: Weil, Courbes algébriques et variétés abéliennes.

3. THE EXPLICIT FORMULAE FOR $\zeta_K(s)$.

For the classical $\zeta(s)$ we have the following result:

Theorem (Explicit Formula): Let f be a function of type (α, β) where $\alpha < -\frac{1}{2}$, $\beta > \frac{1}{2}$ and suppose that there exist $c, \epsilon > 0$ so that for all s satisfying $\alpha \leq \operatorname{Re}(s) \leq \beta$ one has

$$|M(f, s)| < c(1 + |s|)^{-1-\epsilon},$$

where $M(f, s) = \int_{\mathbb{R}} f(x) |x|^{s-1} dx$ (Mellin transform).

Suppose that f is of bounded total variation and that $f(x) = 0$ if $x < 0$. Let

$$\Delta_{\infty}(f) = \lim_{N \rightarrow \infty} \left\{ \int_0^{\infty} f(x) F_N(x) x^{-1} dx - f(1) \log N/\pi \right\}$$

where

$$F_N(x) = \begin{cases} x^{-1/2} (1 - x^{2N}) / |x - x'|, & 0 < x < 1 \\ x^{1/2} (1 - x^{-2N}) / |x - x'|, & 1 < x \\ 0, & x = 1 \end{cases}$$

The limit in this definition exists. Then one has,

$$\Delta_{\infty}(f) + \sum_p \log p \sum_{K \in \mathbb{Z} - \{0\}} p^{-|K|/2} f(p^K)$$

SUM. over the zeros
of $\zeta(s)$ on the
critical strip.

$$= M(f, \frac{1}{2}) + M(f, -\frac{1}{2}) - \sum_{p \in \mathbb{Z}} M(f, p^{-1/2}).$$

Theorem (Explicit Formula - $\zeta_K(s)$): Let $\phi: \mathbb{Z} \rightarrow \mathbb{C}$ be

such that

$$M^d(\phi, z) = \sum_{J \in \mathbb{Z}} \phi(J) z^J$$

(discrete Mellin transform)

converges in the annulus $\{z \in \mathbb{C} : q^{-1/2} \leq |z| \leq q^{1/2}\}$. Then

$$\begin{aligned} & \sum_{w \in \Sigma(K)} \sum_{\ell \in \mathbb{Z}-\{0\}} f(w) q^{-f(w)\ell 1/2} \phi(f(w)\ell) + (2-zg) \phi(0) \\ &= M^d(\phi, q^{1/2}) + M^d(\phi, q^{-1/2}) - \sum_{\alpha \in A} M^d(\phi, \alpha/q^{1/2}). \end{aligned}$$

Proof: Differentiate logarithmically the equation

$$\prod_{w \in \Sigma(K)} (1-u^{f(w)})^{-1} = \left(\prod_{\alpha \in A} (1-\alpha u) \right) (1-u)^{-1} (1-qu)^{-1}$$

$$\begin{aligned} \Rightarrow \sum_{w \in \Sigma(K)} f(w) u^{f(w)} (1-u^{f(w)})^{-1} &= - \sum_{\alpha \in A} \alpha u (1-\alpha u)^{-1} + u (1-u)^{-1} \\ &\quad + qu (1-qu)^{-1}. \end{aligned}$$

power series expansion at each side at 0 and compare the coefficients of u^J for $J > 0$

$$\sum_{\substack{f(w) \mid J \\ f(w) \neq 0}} f(w) = 1 + q^J - \sum_{\alpha \in A} \alpha^J \quad (1)$$

We note that $\sum_{f(w) \mid j} f(w)$ is the number of points of $C(\mathbb{F}_{q^j})$.

Multiply both sides of (1) by $q^{-j/2}$. Then we see the right-hand side of the resulting equation is invariant under replacing j by $-j$. (functional eq.)

So (1) becomes,

$$q^{-|j|/2} \sum_{f(w) \mid j} f(w) = q^{j/2} + q^{-j/2} - \sum_{\alpha \in A} (\alpha/q^{j/2})^j, \quad j \neq 0. \quad (2)$$

Now we multiply (2) by $\phi(k)$ and sum over all k .

4. WEIL'S REFORMULATION OF THE RIEMANN

HYPOTHESIS FOR $\zeta_K(s)$.

- The aim of this section is to give a reformulation of R.H. for $\zeta_K(s)$.

Def.: For ϕ as before we define $\phi^t(x) = \overline{\phi(-x)}$. And for ϕ_1, ϕ_2 as above we define the convolution

$$\phi_1 * \phi_2(x) = \sum_{\substack{j_1, j_2 \\ j_1 + j_2 = x}} \phi_1(j_1) \phi_2(j_2).$$

Proposition: One has that,

$$M^d(\phi_1 * \phi_2, z) = M^d(\phi_1, z) M^d(\phi_2, z).$$

Def: We define a hermitian form R on the vector space of the ϕ above.

For ϕ_1, ϕ_2 in this space,

$$\begin{aligned} R(\phi_1, \phi_2) &= M^d(\psi, q^{1/2}) + M^d(\psi, q^{-1/2}) + (2g-2)\psi(0) \\ &\quad - \sum_{w \in \Sigma(K)} \sum_{l \in \mathbb{Z}-10} f(w) q^{-f(w)|l|/2} \psi(f(w)) \\ &= \sum_{\alpha \in A} M^d(\psi, \alpha/q^{1/2}) ; \quad \psi = \phi_1 * \phi_2^2 \end{aligned}$$

Theorem: The analogue of the Riemann Hypothesis holds (21)
 for $\zeta_K(s)$ if, and only if, the hermitian form R
 is positive semidefinite.

Proof: Suppose that R.H. is true. Then for $\alpha \in A$ we have

$$\begin{aligned} M^d(\phi_1 * \phi_2^2, \alpha/q^{1/2}) &= M^d(\phi_1, \alpha/q^{1/2}) M^d(\phi_2^2, \alpha/q^{1/2}) \\ &= M^d(\phi_1, \alpha/q^{1/2}) \overline{M^d(\phi_2, (\bar{\alpha}/q^{1/2})^{-1})} \\ &= M^d(\phi_1, \alpha/q^{1/2}) \overline{M^d(\phi_2, \alpha/q^{1/2})}. \end{aligned}$$

Thus we have

$$R(\phi_1, \phi_2) = \sum_{\alpha \in A} M^d(\phi_1, \alpha/q^{1/2}) \overline{M^d(\phi_2, \alpha/q^{1/2})}$$

which is clearly positive semidefinite.

$\ker(R)$ is made up of the space of such ϕ that $M^d(\phi, z)$
 has a simple zero at each element of A .

Suppose now that R is positive semidefinite and R.H. does not hold.

Then $\exists \alpha_0 \in A$ s.t. $\alpha_0 := q/\bar{\alpha}_0 \neq \alpha_0$. Since P has
 real coefficients $\alpha_1 \in A$. We choose a polynomial F so
 that $F(\alpha_0) = \sqrt{-1}$, $F(\alpha_1) = -\sqrt{-1}$, $F(\alpha) = 0$ for
 $\alpha \in A - \{\alpha_0, \alpha_1\}$. Then there exists ϕ with $M^d(\phi, z) = F(zq^{1/2})$

From above we have,

$$R(\phi, \phi) = \sum_{\alpha \in A} F(\alpha) \overline{F(q/\bar{\alpha})};$$

in this sum only α_0 and α_1 give a non-zero contribution and each of these contributes -1. Thus $R(\phi, \phi) = -2$ \Rightarrow

■

\rightsquigarrow Analogous statement holds for $S(s)$ - pg 4-5.

with

$$\begin{aligned} R(\phi_1, \phi_2) &= M(\psi, \frac{1}{2}) + M(\psi, -\frac{1}{2}) - \Delta_{\infty}(\psi) \\ &\quad - \sum_p \log p \sum_{K \in \mathbb{Z}-\{0\}} p^{-|K|/2} \psi(p^K) \\ &= \sum_s M(\psi, s - \frac{1}{2}) \end{aligned}$$

Theorem: The R.H. holds $\Leftrightarrow R$ is positive semidefinite.

\rightarrow class Dir. L-functions

\rightarrow Artin L-functions $\xrightarrow{\text{?}}$ these L-function are holomorphic
(Artin Hypothesis).

5. SKETCH OF THE PROOF OF RIEMANN

HYPOTHESIS FOR $\zeta_K(s)$.

* just a sketch since the proof depend on deep considerations
 from Algebraic Geometry.

- Field K we introduce a curve C over \mathbb{F}_q .

- $\bar{K}_0 = \overline{\mathbb{F}_q} \rightarrow \bar{C}$ curve.

(algebraic closure)

- Field of functions of $\bar{C} = K \otimes_{\mathbb{F}_q} \bar{\mathbb{F}_q} =: \bar{K}$

$$F_r : \bar{K} \rightarrow \bar{K} \quad \begin{matrix} \text{gives} \\ \Rightarrow \\ \text{rise} \end{matrix} \quad F_r^J : \bar{C}(\bar{K}_0) \rightarrow \bar{C}(\bar{K}_0)$$

$$u \mapsto u^{q^J}$$

Frobenius Maps

- The fixed points of $F_r^J : \bar{C}(\bar{K}_0) \rightarrow \bar{C}(\bar{K}_0)$

are the points in $C(\mathbb{F}_{q^J})$, since $\mathbb{F}_{q^J} = \{x \in \mathbb{F}_q : x^{q^J} = x\}$.

$$\Rightarrow \text{So number of fixed points is : } \sum_{f(w)=w} f(w) = 1 + q^J - \sum_{\alpha \in A} \alpha^J.$$

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\Rightarrow Now WE NEED COHOMOLOGY THEORY !

- Let ℓ be a prime s.t. $\ell \nmid q$. Then we define the cohomology groups $H^*(\bar{C}, \mathbb{Q}_\ell)$ (coefficients in \mathbb{Q}_ℓ).
- For a curve C of genus g we have $H^J(\bar{C}, \mathbb{Q}_\ell) = 0$ for $J > 2$.
- $H^0(\bar{C}, \mathbb{Q}_\ell)$ and $H^2(\bar{C}, \mathbb{Q}_\ell)$ are one-dimensional
- $H^1(\bar{C}, \mathbb{Q}_\ell)$ is $2g$ -dimensional
- As F_r is of degree $q \Rightarrow$ induced on $H^0(\bar{C}, \mathbb{Q}_\ell)$ is the identity.
- " on $H^2(\bar{C}, \mathbb{Q}_\ell)$ is multiplication by $\frac{1}{q}$.
- By the Lefschetz Fixed-Point Theorem the number of fixed points of F_r^J on $\bar{C}(\bar{\mathbb{F}}_q)$ is

$$\begin{aligned} & \text{Tr} \{ F_r^{*J} | H^0(\bar{C}, \mathbb{Q}_\ell) \} - \text{Tr} \{ F_r^{*J} | H^1(\bar{C}, \mathbb{Q}_\ell) \} \\ & + \text{Tr} \{ F_r^{*J} | H^2(\bar{C}, \mathbb{Q}_\ell) \} . \end{aligned}$$

$1 + q^J$

$$\begin{aligned} \Rightarrow \text{Tr} \{ F_r^{*J} | H^1(\bar{C}, \mathbb{Q}_\ell) \} &= \\ &= \sum_{\alpha \in A} \alpha^J \end{aligned}$$

- α are the eigenvalues of $F_{\ell^*}|H^1(\bar{C}, \mathbb{Q}_{\ell})$.
→ ℓ -adic interpretation

- Let $\phi: \mathbb{Z} \rightarrow \mathbb{Z}$ taking the value 0 at all but a finite number of arguments.

$$\Rightarrow \phi(F_{\ell^*}) := \sum_{k \in \mathbb{Z}} \phi(k) F_{\ell^{*k}}$$

↳ exists as an element of $\text{End}\{H^1(\bar{C}, \mathbb{Q}_{\ell})\}$.

ϕ^2 as before.

\Rightarrow Follows from past theorem that

$$\begin{array}{l} \text{R.H. for } \zeta_k(s) \\ \text{holds} \end{array} \iff \text{Tr} \left\{ \phi(q^{-\frac{s}{2}} F_{\ell^*}) * \phi(q^{-\frac{s}{2}} F_{\ell^*})^2 \right\} \geq 0$$

- $\text{End}\{H^1(C, \mathbb{Q}_{\ell})\} \simeq A \otimes_{\mathbb{Z}} \mathbb{Q}_{\ell}$, A is a special algebra defined over \mathbb{Z} .

when we consider the $\text{End}\{J(\bar{C})\}$

$J(\bar{C})$ - Jacobian variety.

- Frobenius map induces an element of $\text{End}\{\mathcal{J}(\bar{C})\}$ which is invertible in $\text{End}\{\mathcal{J}(\bar{C})\} \otimes \mathbb{Q}$.

- Rosati involution on $\text{End}\{\mathcal{J}(\bar{C})\}$

$$\alpha \mapsto \alpha^t$$

$$\Rightarrow F_r F_r^t = q_L.$$

\Rightarrow R.H. holds
for $\zeta_K(s)$

$$\iff \text{Tr}\{\phi(q^{-\nu_2} F_r^*) * \phi(q^{-\nu_2} F_r^*)^t\} \geq 0$$

↓ ↓
Castelnuovo
Positivity
(true)

$$\text{Tr}(\xi \xi^t) \geq 0$$

6. TRYING TO APPLY WEIL'S PROOF FOR $\zeta(s)$.

- We hope that there would be an analogue of the algebra A with involution.

$$F : \mathbb{R}_+^* \rightarrow A^* \quad \text{homomorphism.}$$

- We hope that A would be endowed with a trace map

$$\text{Tr} : A_1 \rightarrow \mathbb{R} \quad A_1 \subseteq A \quad \text{dense subset.}$$

- Also that

$$\sum_{\rho} M(\phi, \rho - \frac{1}{2}) = \text{Tr} \left\{ \int_{\mathbb{R}_+^*} \phi(t) F(t) t^{-1} dt \right\}$$

holds in $A \otimes \mathbb{C}$. (ϕ satisfies conditions of past theorem and exist).

- The involution in A , so that if $aa^t \in A_1 \Rightarrow \text{Tr}(aa^t) \geq 0$
 $a \mapsto a^t$
- If we also had that $F(t)^2 = F(t^{-1}) \Rightarrow$ R.H. holds.

- No idea how to construct A.
- analogue of the Lefschetz Fixed-Point ?
- " Rosati involution and Castelnuovo positivity ?
- Cohomology for the classical case ?
- Is there an analogue of Frobenius map in the classical case ?

FINAL IDEA :

THE ALGEBRA A SHALL BE EXTRACTED
FROM THE EXPLICIT FORMULA!

THANK FOR YOUR
ATTENTION!!!