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Dear Brian,

Your suggestions in the AIM workshop on how to tackle the Riemann Hypothesis for curves over finite fields through analytic methods led me to revisit the Li's criterion for the Riemann Hypothesis. While I still can not offer you a proof of the Riemann Hypothesis for curves through the use of analytical methods (I hope we can do this by using Levinson's method or other analytic approach), this letter is intended to answer two questions that may be of some interest to you:

1. What is the analogue of Li's criterion for zeta functions of curves over finite fields?

2. There are any geometric interpretation for the Li's coefficients?

Note that in your paper on the Riemann Hypothesis [1] , you say: "It would be interesting to find an interpretation (geometric?) for these λ_n (the Li's coefficients) or perhaps those associated with a different L-function, to make their positivity transparent." The main aim of this letter is to answer your question by giving a geometric interpretation for the Li's coefficients of L–functions associated to curves over finite fields. The second assertion about make the positivity of the Li's coefficients transparent is under investigation by myself (my guess here is that the positivity of the Li's coefficients for function fields are connected with the Castelnuovo-Weil Positivity, one of the main tools used to prove the RH for curves over $\mathbb{F}_q$).

Introduction and Some Background in Function Fields

In his paper, Li [2] proved that the Riemann Hypothesis is equivalent to the positivity of a sequence of real numbers. Below, we quickly review what is the Li's criterion for the Riemann Hypohtesis for $\zeta(s)$ and also we present some basic background of zeta functions over function fields.

Let $\{\lambda_n\}$ be the sequence of numbers given by

$$\lambda_n = \sum_{\rho} \left[1 - \left(1 - \frac{1}{\rho}\right)^n\right], \qquad \text{for all } n \in \mathbb{N} \tag{1}$$

where the sum above runs over the non–trivial zeros ρ of the Riemann zeta-function $\zeta(s)$. These same coefficients can be expressed in terms of the Riemann ξ–function

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s). \tag{2}$$

The coefficients λ_n occur in the Laurent series of $\frac{\xi'}{\xi}$ at $s=0$. Indeed,

$$\frac{d}{dt}\log\xi\left(\frac{t}{t-1}\right) = \frac{-1}{(1-t)^2}\frac{\xi'}{\xi}\left(\frac{t}{t-1}\right) = \sum_{n=0}^{\infty}\lambda_{n+1}t^n, \tag{3}$$

i.e., the *Li coefficients* can be written as

$$\lambda_n = \frac{1}{(n-1)!}\frac{d^n}{ds^n}[s^{n-1}\log\xi(s)]_{s=1}. \tag{4}$$

In 1997, Li [2] proved the following theorem

Theorem 1 (Li [2], 1997)**.** *A necessary and sufficient condition for the nontrivial zeros of the Riemann zeta function to lie on the critical line is that λ_n is non–negative for every positive integer n.*

Further, in 1999, Bombieri and Lagarias [3] presented an extension of this criteria and gave a formula for λ_n using the Weil explicit formulae. One of the main results proved by Bombieri and Lagarias which we will use later in this letter is presented below.

Theorem 2 (Bombieri and Lagarias [3], 1999)**.** *Let $\mathfrak{R}$ be a multi–set of complex numbers ρ such that*

(i) $0, 1 \notin \mathfrak{R}$
(ii) *if $\rho \in \mathfrak{R}$ then $1-\rho$ and $\overline{\rho} \in \mathfrak{R}$ with same multiplicity as ρ.*
(iii)

$$\sum_{\rho} \frac{(1+|Re(\rho)|)}{(1+|\rho|)^2} < \infty. \tag{5}$$

Then the following are equivalents

(a) $Re(\rho) = \frac{1}{2}$ *for all* $\rho \in \mathfrak{R}$.
(b) $\lambda_n = \sum_{\rho} [1-(1-1/\rho)^n] \geq 0$ *for* $n = 1, 2, \ldots$.

In this letter we extend the Li criterion for the Riemann Hypothesis for function fields over finite fields. Before we do this we need a few basic facts of zeta functions over function fields.

Consider K to be a global function field in one variable with a finite constant field $\mathbb{F}_q$ with q elements. Let us denote by C_K the correspondig curve associated to the function field K of genus g over $\mathbb{F}_q$. Then the zeta-function of K is defined by

$$\zeta_K(s) = \sum_{\substack{A \in \mathcal{D}_K \\ A \geq 0}} \frac{1}{NA^s} \qquad \text{Re}(s) > 1, \tag{6}$$

where $\mathcal{D}_K$ denotes the additive group of divisors of K (the free abelian group generated by the primes in K). Typically a divisor $D \in \mathcal{D}_K$ looks like $D = \sum_P a(P)P$ where P is a prime in K and the $a(P)$'s are integer coefficients. We say that $D \geq 0$ if $D \in \mathcal{D}_K$ and $a(P) \geq 0$ for all P. If $A \in \mathcal{D}_K$ we define the norm of A to be $NA = q^{\deg(A)}$ where $\deg(A) = \sum_P a(P)\deg(P)$. We have that $\deg(P)$ is defined to be the dimension of R/P over $\mathbb{F}_q$ where R is a discrete valuation ring in K with maximal idel P such that $\mathbb{F}_q \subset R$ and the quotient field of R is equal to K. For more details see [4, Chapter 5].

The zeta function of K can be rewritten as

$$Z_K(T) = \sum_{n=1}^{\infty} b_n T^n, \tag{7}$$

where $T = q^{-s}$ and $b_n = \#\{D \in \mathcal{D}_K : D \geq 0, \deg(D) = n\}$. Using the multiplicativity of the norm NA and the fact that $\mathcal{D}_K$ is a free abelian group on the set of primes of K we see, at least formally, that

$$\zeta_K(s) = \prod_{\substack{P \\ \text{prime divisor in } K}} \left(1 - \frac{1}{NP^s}\right)^{-1} \qquad \text{for Re}(s) > 1, \tag{8}$$

One can regard the zeta–function of K as an analogue of the classical Riemann zeta function and it can be written as

$$\zeta_K(s) = Z_K(T = q^{-s}). \tag{9}$$

By Weil [5] we have that $\zeta_K(s)$ satisfies the following properties:

(i) [Rationality] $\zeta_K(s)$ is a rational function

$$Z_K(T) = \frac{L_K(T)}{(1-T)(1-qT)}, \tag{10}$$

where $L(T) \in \mathbb{Z}[T]$ is a polynomial of degree $2g$.

(ii) [Functional Equation] We have

$$Z_K(q^{-s}) = q^{g-1}q^{-s}(2g-2)Z_K\left(\frac{1}{q^{1-s}}\right), \tag{11}$$

or making use of the s variable we have

$$q^{s(g-1)}\zeta_K(s) = q^{(1-s)(g-1)}\zeta_K(1-s), \tag{12}$$

for all $s \in \mathbb{C}$.

(iii) [Riemann Hypothesis for Function Fields] All the roots of $\zeta_K(s)$ lie on the line Re$(s) = 1/2$. Equivalently, the inverse roots of $L_K(T)$ all have absolute value $\sqrt{q}$.

From (10) we see that $\zeta_K(s)$ is a meromorphic function in the whole complex plane $\mathbb{C}$ with simple poles at $1 + k\frac{2\pi i}{\log(q)}$ and $k\frac{2\pi i}{\log(q)}$ for $k \in \mathbb{Z}$.

The polynomial

$$L_K(T) = (1-T)(1-qT)Z_K(T) \tag{13}$$

is called the L–polynomial of the curve C_K and from (11) we can see that $L_K(T)$ satisfies the functional equation

$$L_K(T) = q^g T^{2g} L_K\left(\frac{1}{qT}\right). \tag{14}$$

We write

$$L_K(T)\sum_{n=0}^{2g} a_n T^n, \tag{15}$$

where $a_i \in \mathbb{Z}$, $a_0 = 1$, $a_{2g} = q^g$ and the functional equation (Poincaré duality) gives us that $a_{2g-i} = q^{g-i}a_i$ for $0 \leq i \leq g$.

Since the coefficients of the polynomial $L_K(T)$ are in $\mathbb{Z}$ we can factor this polynomial over $\mathbb{C}$ as

$$L_K(T) = \prod_{i=1}^{2g}(1-\alpha_i T), \tag{16}$$

where the α_i's are the inverse roots of $L_K(T)$ and they can be arranged such as $\alpha_i\alpha_{g+i} = q$ holds for $i = 1, \ldots, g$. For more details see [4].

The Li Criterion for Riemann Hypothesis for Curves over $\mathbb{F}_q$

Consider the function

$$\xi_K(s) = (1-q^{-s})(1-q^{1-s})\Lambda_K(s), \tag{17}$$

where

$$\Lambda_K(s) = q^{(g-1)s}\zeta_K(s) \tag{18}$$

and satisfies the functional equation

$$\Lambda_K(s) = \Lambda_K(1-s). \tag{19}$$

This function is an entire function of order one. By analogy we define the Li coefficients for the function field K by

$$\lambda_K(n) = \sum_{\rho_K} \left[1 - \left(1 - \frac{1}{\rho_K} \right)^n \right], \qquad n \geq 1, \tag{20}$$

where the sum runs over the non–trivial zeros ρ_K of $\zeta_K(s)$. In the same way as in the classical case the coefficients $\lambda_K(n)$ occur also in the Taylor expansion of $\frac{\xi_K'}{\xi_K}$ at $s = 0$, i.e.,

$$\frac{d}{dt} \log \xi_K \left(\frac{t}{t-1} \right) = \sum_{n=0}^{\infty} \lambda_K(n+1) t^n. \tag{21}$$

The following proposition presents an explicit formula for the Li coefficients $\lambda_K(n)$.

Proposition 1. *For all $n \geq 1$ we have that,*

$$\lambda_K(n) = -n \left\{ \sum_{i=1}^{2g} \sum_{r=1}^{\infty} \frac{(\alpha_i)^r}{r} \left[\sum_{k=0}^{n-1} \binom{n-1}{k} \frac{(\log q^r)^{k+1}}{(k+1)!} \right] \right\}. \tag{22}$$

Proof. From the definition of $\xi_K(s)$, we have that

$$\begin{aligned} \log \xi_K \left(\frac{t}{t-1} \right) &= \log L_K(q^{-\frac{t}{t-1}}) \\ &= \log \prod_{i=1}^{2g} (1 - \alpha_i q^{-\frac{t}{t-1}}) \\ &= -\sum_{i=1}^{2g} \sum_{r=1}^{\infty} \frac{(\alpha_i)^r}{r} q^{-\frac{rt}{t-1}} \\ &= -\sum_{i=1}^{2g} \sum_{r=1}^{\infty} \frac{(\alpha_i)^r}{r} \left(1 + \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n-1} \binom{n-1}{k} \frac{(\log q^r)^{k+1}}{(k+1)!} \right) t^n \right). \end{aligned} \tag{23}$$

From (21) the proposition follows. □

The following result it is not important for the main discussion of this letter. So I will not present the proof here. (it follows from the class number h_K associated to the function field K and from Poincaré–duality).

Proposition 2. *We have that $\lambda_K(1) \geq 0$ for all function fields K, i.e., for all curves C_K over $\mathbb{F}_q$.*

The next result answers question (1) about the Li criterion for function fields.

Theorem 3. *The non–trivial zeros of $\zeta_K(s)$ lie on the line Re$(s) = \frac{1}{2}$ if and only if $\lambda_K(n) \geq 0$ for all $n \in \mathbb{N}$.*

Proof. Let

$$\mathcal{Z}(\zeta_K(s)) = \left\{ \rho_K = \frac{1}{2} \pm i \frac{\theta_j}{\log(q)} + i \frac{2k\pi}{\log(q)}, j \in \ 1, \ldots, g \ , k \in \mathbb{Z} \right\}, \tag{24}$$

where $\theta_1, \theta_2, \ldots, \theta_g \in [0, 2\pi[$ are the angles of the roots of $L_K(T)$. Moreover, the multi–set $\mathcal{Z}(\zeta_K(s))$ is invariant under the symmetry $\rho_K \mapsto 1 - \overline{\rho_K}$. Therefore the multi–set $\mathcal{Z}(\zeta_K(s))$ satisfies the hypothesis of Theorem 2. Therefore the results follows. □

A Geometric Interpretation for $\lambda_K(n)$

In this section I present a geometric interpretation for the Li coefficients associated to the zeta function $\zeta_K(s)$ of a function field K. In this way I hope have clarified your assertion from the paper [1].

Theorem 4. *The following conditions are equivalent*

(i) The zeros of $\zeta_K(s)$ lie on the line Re$(s) = \frac{1}{2}$.
(ii) $|\lambda_K(n)| \leq 2gq^{n/2}$ for all $n \in \mathbb{N}$.

Proof. Assumes that the zeros of $\zeta_K(s)$ lie on the line Re$(s) = \frac{1}{2}$, then we can conclude that the roots of $L_K(T)$ are of absolute value $q^{-1/2}$. Note that

$$\log(L_K(T)) = \sum_{n=1}^{\infty} \frac{\lambda_K(n)}{n} T^n. \tag{25}$$

Taking the logarithm of (16) we have that

$$\sum_{n=1}^{\infty} \frac{\lambda_K(n)}{n} T^n = -\sum_{i=1}^{2g} \sum_{n=1}^{\infty} \frac{(\alpha_i)^n}{n} T^n, \tag{26}$$

where $|\alpha_i| = q^{1/2}$. Then,

$$\lambda_K(n) = -\sum_{i=1}^{2g} \alpha_i^n, \tag{27}$$

for all $n \geq 1$. Using that $|\alpha_i| = q^{1/2}$ we obtain that $|\lambda_K(n)| \leq 2gq^{n/2}$.

Conversely, if $|\lambda_K(n)| \leq 2gq^{n/2}$ for all $n \in \mathbb{N}$, then $R \geq q^{-1/2}$, where R is the radius of convergence of $\sum_{n=1}^{\infty} \frac{\lambda_K(n)}{n} T^n$. Indeed, if $T = \varepsilon q^{-1/2}$ with $|\varepsilon| < 1$, then $\sum_{n=1}^{\infty} \left|\frac{\lambda_K(n)}{n}\right| |T|^n \leq 2g \log(1-\varepsilon)$. Therefore, from (25), $L_K(T)$ has no roots in the disc $|T| < q^{-1/2}$ (i.e., $|\alpha_i| \leq q^{1/2}$).

Using the fact that $\prod_{i=1}^{2g} \alpha_i = a_{2g} = q^g$, we obtain the desired result. ☐

The zeta function $Z_K(T)$ encodes information about the rational points on the curve C_K. If we call $N_n = |C_K(\mathbb{F}_{q^n})|$ to be the number of $\mathbb{F}_{q^n}$–rational points on the curve C_K corresponding to the function field K over $\mathbb{F}_q$ then by definition the zeta function of the curve C_K is given by

$$Z_K(T) = \exp\left(\sum_{n=1}^{\infty} \frac{N_n}{n} T^n\right). \tag{28}$$

Theorem 5. *The zeros of $\zeta_K(s)$ lie on the line Re$(s) = \frac{1}{2}$ if and only if*

$$|N_n - (q^n + 1)| \leq 2gq^{n/2}, \tag{29}$$

for all $n \geq 1$.

Proof. From (10), we have

$$\log(L_K(T)) = \log(1-T) + \log(1-qT) + \log(Z_K(T)). \tag{30}$$

Hence from (28) and (25), we obtain

$$\sum_{n=1}^{\infty} \frac{\lambda_K(n)}{n} T^n = -\sum_{n=1}^{\infty} \frac{T^n}{n} - \sum_{n=1}^{\infty} \frac{q^n T^n}{n} + \sum_{n=1}^{\infty} N_n \frac{T^n}{n}. \tag{31}$$

Therefore

$$\log(L_K(T)) = \sum_{n=1}^{\infty} \left(\frac{N_n - q^n - 1}{n}\right) T^n. \tag{32}$$

Then, from Teorem 4 the result follows. □

The geometric interpretation for the $\lambda_K(n)$ is that

$$\lambda_k(n) = N_n - q^n - 1, \tag{33}$$

i.e., the Li coefficients for the function field K is a function on the number of rational points N_n of the curve C_K.

Consider $C_D : y^2 = D(T)$ be an hyperelliptic curve of genus g over $\mathbb{F}_q$ where $D(T)$ is a square–free monic polynomial in $\mathbb{F}_q[T]$. In this case

$$\begin{aligned} L_{C_D}(T) &= \det(I - T\sqrt{q}\Theta_{C_D}) \\ &= \det(I - T\mathrm{Fr}_q|H^1(C_D)), \end{aligned} \tag{34}$$

where $\Theta_{C_D} \in USp(2g)$ and $H^1(C_D)$ denotes the first cohomology of the curve C_D. From (27) we have conclude that

$$\lambda_K(n) = -\sum_{i=1}^{2g} \alpha_i^n = -\mathrm{tr}\Theta_{C_D}^n. \tag{35}$$

And therefore equation (35) provides an interpretation for $\lambda_K(n)$ in terms of the trace of powers of Θ_{C_D} or if you prefer in terms of the Frobenius acting on the $H^1(C_D)$ cohomology.

To conclude, we not only provided a geometric interpretation for the $\lambda_K(n)$, but also a cohomological interpretation and an interpretation in terms of the conjugacy classes associated to the curve C_D.

With best regards,
Julio Andrade

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