Comments on Jon's Calculations about the Averaging Hardy-Littlewood.

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1-) Your main result is an asymptotic formula for

$$\sum_{\deg(K)=n} \sigma(K) \qquad \text{as} \qquad q \to \infty$$

as said in pg 4. in your notes.

But the most natural average values for functions defined on the set of monic polynomials is to consider the limit as $n \to \infty$ [pg 16 - Rosen].

2-) Your final answer is

$$\sum_{\deg(K)=n} \sigma(K) = q^n - \frac{1}{q-1}$$

If we set $|K| = q^n = x$ the result can be rewritten

$$\sum_{\substack{K \text{ monic} \\ |K| = x}} \sigma(K) = x - \frac{1}{q-1}$$

It is known that the number $q-1$ often occurs where the number 2 occurs in ordinary number theory. This is due the fact that the order of $\mathbb{Z}^*$ is 2 and the order of $\mathbb{F}_q^*$ is $q-1$.

So comparing the Standard Case

$$\sum_{h \leq x} c(h) \sim x - \frac{1}{2} \log x$$

with the Function Field case

$$\sum_{\substack{K \text{ monic} \\ \deg K = n \\ \text{or} \\ |K| = x}} \sigma(K) = x - \frac{1}{q-1} \qquad (1)$$

Thus the only difference that I do not understand and I was expecting is to have n multiplying the term $\frac{1}{q-1}$ in (1). But I have checked your contour integral and seems completely right, so for the function field case seems to be the case that the corresponding term to $\log x$ does not appear.

Sincerely,

Julio.

Averaging Hardy-Littlewood

1. The Standard Case

$$C(h) = 0 \quad \text{if } h \text{ is odd}$$

$$= \prod_{p \nmid h} \frac{p(p-2)}{(p-1)^2} \prod_{p \mid h} \frac{p}{p-1}$$

$$= 2 \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{\substack{p \mid h \\ p>2}} \frac{p-1}{p-2} \quad \text{if } h \text{ is even}$$

We seek an asymptotic estimate for

$$\sum_{h=1}^{X} C(h) = \underbrace{2 \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right)}_{\alpha} \sum_{h=1}^{X} \alpha(h) \tag{1}$$

Consider

$$F(s) = \sum_{h=1}^{\infty} \frac{\alpha(h)}{h^s}$$

Then

$$\sum_{h=1}^{X} \alpha(h) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) \frac{X^s}{s} \, ds \tag{2}$$

and a <u>double pole</u> at $s=0$. As $X \to \infty$ the leading order contribution of the pole at $s=0$ to (2) is

$$\frac{1}{2^0}\left(1-\frac{1}{2}\right)\zeta(0)\prod_{p>2}\left(1+\frac{2}{p(p-2)}-\frac{1}{p(p-2)}\right)\log X$$

$$= -\frac{1}{2}\frac{1}{2}\log X \qquad (5)$$

$\zeta(0) = -\frac{1}{2}$

So combining (4) and (5)

$$\sum_{h=1}^{X} c(h) \sim X - \frac{1}{2}\log X \qquad (6)$$

'on average' $c(h)$ behaves like $1-\frac{1}{2|h|}$.

2. The Function Field Case

$$\sigma(k) = \prod_{P}\left(1-\frac{1}{|P|}\right)^{-2}\prod_{P|k}\left(1-\frac{1}{|P|}\right)\prod_{P\nmid k}\left(1-\frac{2}{|P|}\right)$$

$$= \prod_{P}\left(1-\frac{1}{(|P|-1)^2}\right)\prod_{P|k}\frac{|P|-1}{|P|-2}$$

where $|P| = q^{\deg P}$

where the contour is a circle small enough so that the sum in the integrand converges.

Now $$\sum_k \sigma(k)\, u^{\deg k} = \tilde{\alpha}_A\, Z_A(u)\, Z_A(u/q) \times \prod_P \left(1 + \frac{2u^{\deg P}}{|P|(|P|-2)} - \frac{|P|^{-1} u^{2\deg P}}{(|P|-2)}\right)$$

$$\prod_P (1-u^{\deg P})^{-1}$$

And using $Z_A(u) = \dfrac{1}{1-qu}$

$$\sum_k \sigma(k)\, u^{\deg k} = \tilde{\alpha}_A \frac{1}{1-qu} \frac{1}{1-u} \prod_P \left(1 + \frac{2u^{\deg P}}{|P|(|P|-2)} - \frac{u^{2\deg P}}{|P|(|P|-2)}\right)$$

so the integrand in (8) has poles at $u = 1/q$, $u = 1$

~~a pole at $u = 1/q$ with residue~~

Expanding the contour ($|u| \to \infty$), the residues are

- at $u = 1/q$ $\quad \tilde{\alpha}_A q^{-1} \dfrac{1}{\tilde{\alpha}_A} q^{n+1} = q^n$ ✓
- at $u = 1$ $\quad \tilde{\alpha}_A \dfrac{1}{1-q} \dfrac{1}{\tilde{\alpha}_A} = \dfrac{1}{1-q}$

This suggests that as $q \to \infty$

$$\sum_{\deg k = n} \sigma(k) = q^n - \frac{1}{q-1}$$