Dear Professor Michael Rubinstein,

In this short note I establish an approximate functional equation for the even case $(2g+2)$ as you have asked in your e-mail from 06/03/2014. I will use the notation of my paper with Keating [1].

Let $D \in \mathbb{F}_q[T]$ be a square-free monic polynomial. Then we have

$$\mathcal{L}(u, \chi_D) = (1-u)^{\lambda} \mathcal{L}^*(u, \chi_D), \tag{1}$$

where

$$\lambda = \begin{cases} 1, & \deg(D) \text{ even} \\ 0, & \text{otherwise} \end{cases}. \tag{2}$$

From now let us assume $D \in \mathcal{H}_{2g+2,q}$, then we have

$$\mathcal{L}(u, \chi_D) = (1-u)\, \mathcal{L}^*(u, \chi_D), \tag{3}$$

where

$$\mathcal{L}^*(u, \chi_D) = \sum_{n=0}^{2g} \sigma_n^*(D)\, u^n. \tag{4}$$

Indeed $\mathcal{L}^*(u, \chi_D) = P_{C_D}(u)$, where $P_{C_D}(u)$ is the numerator of the zeta function associated to the curve $C_D : y^2 = D(T)$. We now present the main result of this note.

<u>Theorem</u>: Let $D \in \mathcal{H}_{2g+2,q}$, then

$$\mathcal{L}(q^{-1/2}, \chi_D) = \sum_{n=0}^{g} \sum_{\deg f = n} \chi_D(f)\, q^{-n/2} + q^{-\frac{(g+1)}{2}} \sum_{n=0}^{g} \sum_{\deg f = n} \chi_D(f)$$
$$+ \sum_{n=0}^{g-1} \sum_{\deg f = n} \chi_D(f)\, q^{-n/2} + q^{-g/2} \sum_{n=0}^{g-1} \sum_{\deg f = n} \chi_D(f). \tag{5}$$

Proof: Write

$$\mathcal{L}^*(u,\chi_D) = \sum_{n=0}^{2g} \sigma_n^*(D)\, u^n. \tag{6}$$

Since $\mathcal{L}(u,\chi_D) = (1-u)\,\mathcal{L}^*(u,\chi_D)$, we have

$$\sigma_n(D) = \begin{cases} \sigma_0^*(D), & \text{if } n=0, \\ \sigma_n^*(D) - \sigma_{n-1}^*(D), & \text{if } 1 \le n \le 2g, \\ -\sigma_{2g}^*(D), & \text{if } n = 2g+1, \end{cases} \tag{7}$$

where $\mathcal{L}(u,\chi_D) = \sum_{n=0}^{2g+1} \sigma_n(D)\, u^n$. From (7) we have

$$\sigma_n^*(D) = \sum_{i=0}^{n} \sigma_i(D) \qquad (0 \le n \le 2g). \tag{8}$$

By substituting $\mathcal{L}^*(u,\chi_D)$ into the functional equation for $P_{C_D}(u)$,

$$\begin{aligned} \sum_{n=0}^{2g} \sigma_n^*(D)\, u^n &= \sum_{n=0}^{2g} \sigma_n^*(D)\, q^{g-n} u^{2g-n} \\ &= \sum_{n=0}^{2g} \sigma_{2g-n}^*(D)\, q^{-g+n} u^n. \end{aligned} \tag{9}$$

Equating coefficients we have,

$$\sigma_n^*(D) = \sigma_{2g-n}^*(D)\, q^{-g+n} \quad \text{or} \quad \sigma_{2g-n}^*(D) = \sigma_n^*(D)\, q^{g-n}, \tag{10}$$

and we can write $\mathcal{L}^*(u,\chi_D)$ as

$$\mathcal{L}^*(u,\chi_D) = \sum_{n=0}^{g} \sigma_n^*(D)\, u^n + q^g u^{2g} \sum_{n=0}^{g-1} \sigma_n^*(D)\, q^{-n} u^{-n}. \tag{11}$$

In particular, we have

$$\mathcal{L}^*(q^{-1/2}, \chi_D) = \sum_{n=0}^{g} \sigma_n^*(D)\, q^{-n/2} + \sum_{n=0}^{g-1} \sigma_n^*(D)\, q^{-n/2}. \qquad (12)$$

By substituting (8) in (12), we have

$$\begin{aligned} \mathcal{L}^*(q^{-1/2}, \chi_D) &= \sum_{n=0}^{g} \sum_{i=n}^{g} q^{-i/2} \sigma_n(D) + \sum_{n=0}^{g-1} \sum_{i=n}^{g-1} q^{-i/2} \sigma_n(D) \qquad (13) \\ &= \sum_{n=0}^{g} \sigma_n(D) \left(\frac{q^{-n/2} - q^{-\left(\frac{g+1}{2}\right)}}{1 - q^{-1/2}} \right) + \sum_{n=0}^{g-1} \sigma_n(D) \left(\frac{q^{-n/2} - q^{-g/2}}{1 - q^{-1/2}} \right). \end{aligned}$$

∎

I think $\mathcal{L}(u, \chi_D)$ is a better object for the recipe for moments. I suppose you can use equation (5) in the recipe.

But regarding the maximum value of $\mathcal{L}^*(u,\chi_D)$ and $\mathcal{L}(u,\chi_D)$, I think that $\mathcal{L}^*(u,\chi_D)$ definitely is a better object to exploit.

With the Best Wishes,

Julio Andrade

References

[1] J.C. Andrade and J.P. Keating, The mean value of $L(1/2,\chi)$ in the hyperelliptic ensemble, Journal of Number Theory (2012).