MEAN VALUES OF DERIVATIVES OF L-FUNCTIONS IN FUNCTION FIELDS: II

JULIO ANDRADE

Abstract. This is the second part of our investigation of mean values of derivatives of L-functions in function fields. In this paper, specifically, we prove several mean values results for the derivatives of L-functions in function fields when the average is taken over all discriminants, i.e., over all monic polynomials of a prescribed degree in $\mathbb{F}_q[T]$. We establish exact formulas for the mean value of the μ-th derivative of L-functions in function fields at the critical point and we compute a few particular examples.

1. Introduction

In the first paper in this series, see [1], we computed the first moment of derivatives of quadratic Dirichlet L-functions in function fields using classical techniques such as character sums, approximate functional equation and the Riemann Hypothesis for curves. In particular, we calculated the full polynomial in the asymptotic expansion of $\sum_{D\in\mathcal{H}} L''(\frac{1}{2},\chi_D)$, with $L(s,\chi_D)$ being the quadratic Dirichlet L-function attached to the quadratic character χ_D where D is monic and square-free and $\mathcal{H}$ is the set of all monic and square-free polynomials of odd degree in $\mathbb{F}_q[T]$. In a subsequent paper, we will show how to generalize the result above and compute the first moment of the μ-th derivative of quadratic Dirichlet L-functions and formulate conjectures for higher moments of derivatives of this family of L-functions.

In this paper, we prove several mean values results for the derivatives of L-functions in function fields when the average is taken over all discriminants, i.e., over all monic polynomials of a prescribed degree. Differently from the first paper, where we computed the first moment for the second derivative, here we are going to compute the first moment for the generic μ-th derivative for different families of L-functions, where $\mu \geq 1$.

This series of papers has two main motivations. The first motivation comes from the study of moments of derivatives of the Riemann zeta function, which was initiated by Ingham [8], and then further developed by the work of Conrey [2], Gonek [5], Conrey, Rubinstein and Snaith [3], Hughes,

Date: August 30, 2017.

2010 *Mathematics Subject Classification.* Primary 11M38; Secondary 11M06, 11G20, 11M50, 14G10.

Key words and phrases. function fields, derivatives of L–functions, moments of L–functions, quadratic Dirichlet L–functions, random matrix theory.

Keating and O'Connell [7], Milinovich and Ng [11, 12], Laurinčikas and Steuding [10] and several other authors. The second motivation is coming from the pioneering work of Hoffstein and Rosen [6] about the study of mean values of Dirichlet L-functions in function fields. See [1] for a summarized account of some of the previous results about moments of derivatives of the Riemann zeta function.

Before we proceed and state the main results of this paper and some previous results on mean values of L-functions in function fields we are going to need a few basic definitions about the theory of L-functions in function fields. See [13] for a detailed exposition about L-functions in function fields.

Let $\mathbb{F}_q$ be a finite field of odd cardinality and we denote $A = \mathbb{F}_q[T]$ to be the polynomial ring over $\mathbb{F}_q$ and $k = \mathbb{F}_q(T)$ to be the rational function field over $\mathbb{F}_q$. The norm of a polynomial $f \in A$ is defined to be $|f| = q^{\deg(f)}$. We now introduce the zeta function of A to be

$$\begin{aligned} \zeta_A(s) &= \sum_{f \text{ monic}} |f|^{-s} \\ &= \frac{1}{1-q^{1-s}}. \end{aligned} \tag{1.1}$$

From now on sums over polynomials will denote sums over monic polynomials unless the contrary is stated.

For m a non-square in A we define the Dirichlet character modulo m using the Jacobi symbol in $\mathbb{F}_q[T]$ by

$$\chi_m(f) = \left(\frac{m}{f}\right), \tag{1.2}$$

for $f \in A$. The Jacobi symbol is defined multiplicatively by the Legendre symbol in $\mathbb{F}_q[T]$ given by

$$\chi_m(P) = \begin{cases} 0 & \text{if } P \mid m \\ 1 & \text{if } P \nmid m \text{ and } m \text{ is a square modulo } P \\ -1 & \text{if } P \nmid m \text{ and } m \text{ is a non-square modulo } P, \end{cases} \tag{1.3}$$

with P a monic and irreducible polynomial in A. From the quadratic reciprocity law in function fields [6, Lemma 0.5] we have that if $a \equiv b (\text{mod } m)$ and $\deg(a) \equiv \deg(b) (\text{mod } 2)$, then $\chi_m(a) = \chi_m(b)$.

We have all the ingredients to define the Dirichlet L-function associated to the character χ_m and we do this now.

$$L(s, \chi_m) = \sum_{f \text{ monic}} \chi_m(f) |f|^{-s}. \tag{1.4}$$

This sum converges for $\text{Re}(s) > 1$, but in fact can be shown that $L(s, \chi_m)$ is a finite sum given by

$$L(s,\chi_m) = \sum_{k=0}^{M-1} S_k(\chi_m) q^{-ks}, \tag{1.5}$$

where the degree of the polynomial m is M and

$$S_k(\chi_m) = \sum_{\substack{\deg(n)=k \\ n \text{ monic}}} \chi_m(n). \tag{1.6}$$

In [6], Hoffstein and Rosen established several mean values results concerning Dirichlet L-functions over function fields. We now present some of their main results that are important and connected with the main results of this note. The first result is about mean values of L-functions with degree odd conductor. Let m be a non-square monic polynomial and $\deg(m) = M$, where $M > 0$ is an odd integer, then they proved that for $s \neq \frac{1}{2}$

$$q^{-M} \sum_{\deg(m)=M} L(s,\chi_m) = \frac{\zeta_A(2s)}{\zeta_A(2s+1)} - \left(1 - \frac{1}{q}\right) (q^{1-2s})^{\frac{M+1}{2}} \zeta_A(2s). \tag{1.7}$$

And at the critical point they established that

$$q^{-M} \sum_{\deg(m)=M} L(\tfrac{1}{2},\chi_m) = 1 + \left(1 - \frac{1}{q}\right) \left(\frac{M-1}{2}\right). \tag{1.8}$$

In the same paper [6], they proved that, if $M > 0$ is even and the following sums are over all non-square monic polynomials of degree M, and $s \neq \frac{1}{2}$ or 1 then

$$\begin{aligned} q^{-M} \sum L(s,\chi_m) = & \frac{\zeta_A(2s)}{\zeta_A(2s+1)} - \left(1 - \frac{1}{q}\right) (q^{1-2s})^{\frac{M}{2}} \zeta_A(2s) \\ & - q^{-\frac{M}{2}} \left(\frac{\zeta_A(2s)}{\zeta_A(2s+1)} - \left(1 - \frac{1}{q}\right) (q^{1-s})^M \zeta_A(s) \right). \end{aligned} \tag{1.9}$$

And for $s = 1$ they established that

$$q^{-M} \sum L(1,\chi_m) = \frac{\zeta_A(2)}{\zeta_A(3)} - q^{-\frac{M}{2}} \left(2 + \left(1 - \frac{1}{q}\right)(M-1)\right). \tag{1.10}$$

Another variant of mean values of L-functions over function fields is to consider $M > 0$ be an even integer, and let $\gamma \in \mathbb{F}_q^*$ be a non-square constant. In this particular case, for $s \neq \frac{1}{2}$ and the sum being taken over all non-square monic polynomials of degree M, Hoffstein and Rosen [6] proved that

$$q^{-M}\sum L(s,\chi_{\gamma m}) = \frac{\zeta_A(2s)}{\zeta_A(2s+1)} - \left(1-\frac{1}{q}\right)(q^{1-2s})^{\frac{M}{2}}\zeta_A(2s)$$
$$- q^{-\frac{M}{2}}\left(\frac{1+q^{-s}}{1+q^{1-s}} - \left(1-\frac{1}{q}\right)\frac{(q^{1-s})^M}{1+q^{1-s}}\right). \tag{1.11}$$

We are now in a position to state the main results of this paper. The following theorems can be seen as an extension and generalization of those previous results due to Hoffstein and Rosen. The techniques used to prove the next theorems are in essence the same as those used by Hoffstein and Rosen in their beautiful paper but in our case, the calculations are more involving.

Theorem 1.1. *Let $M > 0$ be an odd integer and $\mu \geq 1$ an integer. The following sum is over all monic m of degree M.*

$$\sum L^{(\mu)}(\tfrac{1}{2},\chi_m) = \frac{(-2\log q)^{\mu}}{\mu+1}\left(1-\frac{1}{q}\right)q^M\sum_{l=0}^{\mu}(-1)^l\binom{\mu+1}{l}B_l\left(\frac{M-1}{2}\right)^{\mu+1-l}, \tag{1.12}$$

where B_l are the Bernoulli numbers and $B_1 = -\frac{1}{2}$.

As M goes to infinity, the dominating term on the RHS of (1.12) occurs when $l = 0$, from which we have the following corollary.

Corollary 1.1. *With the same hypotheses as in the theorem,*

$$\sum L^{(\mu)}(\tfrac{1}{2},\chi_m) \sim \frac{(-2\log q)^{\mu}}{\mu+1}\left(1-\frac{1}{q}\right)q^M\left(\frac{M-1}{2}\right)^{\mu+1}, \tag{1.13}$$

as $M \to \infty$.

As an example of Theorem 1.1 we have the following formulas for the first and the second derivative at the central point.

$$\sum L^{(1)}(\tfrac{1}{2},\chi_m) = -\frac{1}{4}(\log q)(-1+q)(-1+M^2)q^{M-1}.$$

$$\sum L^{(2)}(\tfrac{1}{2},\chi_m) = \frac{1}{6}(\log q)^2(-1+q)M(M^2-1)q^{M-1}.$$

The next results are more involving and are about mean values of derivatives of L-functions over function fields when the conductor of the character has even degree. Our first result in this direction is

Theorem 1.2. *Let $M > 0$ be an even integer and $\mu \geq 1$ also an integer. The following sum is over all non-square monic polynomials of degree M.*

$$\sum L^{(\mu)}(\tfrac{1}{2},\chi_m) = \frac{(-2\log q)^{\mu}}{\mu+1}\left(1-\frac{1}{q}\right)q^M\sum_{l=0}^{\mu}(-1)^l B_l\binom{\mu+1}{l}\left(\frac{M-2}{2}\right)^{\mu+1-l}$$

$$-(-\log q)^{\mu}\left(1-\frac{1}{q}\right)q^{\frac{M}{2}}\left(q^{-\frac{M}{2}}\Phi(\sqrt{q},-\mu,M)+\mathrm{Li}_{-\mu}(\sqrt{q})\right), \tag{1.14}$$

where B_l are, as before, the Bernoulli numbers, $\Phi(z,s,\alpha)$ is the Lerch transcendent function given by

$$\Phi(z,s,\alpha) = \sum_{n=0}^{\infty}\frac{z^n}{(n+\alpha)^s}, \tag{1.15}$$

and $\mathrm{Li}_s(z)$ is the polylogarithm function given by

$$\mathrm{Li}_s(z) = \sum_{k=1}^{\infty}\frac{z^k}{k^s}. \tag{1.16}$$

Remark 1.2. *As will be shown in Section 4* $\left(q^{-\frac{M}{2}}\Phi(\sqrt{q},-\mu,M)+\mathrm{Li}_{-\mu}(\sqrt{q})\right)$ *is a finite sum that can be explicitly computed.*

The next result is a simple corollary of the above theorem.

Corollary 1.3. *With the same hypotheses as in the previous theorem we have that*

$$\sum L^{(\mu)}(\tfrac{1}{2},\chi_m) \sim \frac{(-2\log q)^{\mu}}{\mu+1}\left(1-\frac{1}{q}\right)q^M\left(\frac{M-2}{2}\right)^{\mu+1}, \tag{1.17}$$

as $M\to\infty$.

As an example of Theorem 1.2 we have the following formulas for the first and the second derivative at the central point.

$$\sum L^{(1)}(\tfrac{1}{2},\chi_m) = -\frac{(1+\sqrt{q})}{4(\sqrt{q}-1)q}(\log q)$$
$$\times\left(M^2(\sqrt{q}-1)^2q^M-4(\sqrt{q}+q^{\frac{M+1}{2}})+M(-4+4\sqrt{q}-2q^M+4q^{M+\frac{1}{2}}-2q^{M+1})\right).$$

$$\sum L^{(2)}(\tfrac{1}{2},\chi_m) = \frac{(-1+q)(\log q)^2}{6q}$$
$$\times\left(M(M^2-3M+2)q^M+\frac{6\left(M^2(\sqrt{q}-1)^2+\sqrt{q}+q+q^{\frac{M}{2}+1}+q^{\frac{M+1}{2}}-2M(-\sqrt{q}+q)\right)}{(\sqrt{q}-1)^3}\right).$$

Our last theorem explores the case when we sum over γm, $\gamma \in \mathbb{F}_q^*$ a fixed non-square, and m varying over all non-square monics of degree M. The case γm_1^2, where $m = m_0 m_1^2$ with m_0 square-free, is avoided because then $k(\sqrt{\gamma m_1^2}) = k(\sqrt{\gamma})$ is the constant field extension of k, and we wish to exclude the constant field extension since its arithmetic is uninteresting for our purposes.

Theorem 1.3. *Let $M > 0$ be even, $\mu \geq 1$ an integer and $\gamma \in \mathbb{F}_q^*$ a non-square. If we sum over all non-square monics m of degree M, then*

$$\sum L^{(\mu)}(\tfrac{1}{2}, \chi_{\gamma m}) = \frac{(-2\log q)^{\mu}}{\mu+1}\left(1-\frac{1}{q}\right) q^M \sum_{l=0}^{\mu} (-1)^l B_l \binom{\mu+1}{l} \left(\frac{M-2}{2}\right)^{\mu+1-l}$$
$$-(-\log q)^{\mu}\left(1-\frac{1}{q}\right) q^{\frac{M}{2}} \left(-(-\sqrt{q})^M \Phi(-\sqrt{q}, -\mu, M) + \mathrm{Li}_{-\mu}(-\sqrt{q})\right). \tag{1.18}$$

Remark 1.4. *Note that $\left(-(-\sqrt{q})^M \Phi(-\sqrt{q}, -\mu, M) + \mathrm{Li}_{-\mu}(-\sqrt{q})\right)$ is a finite sum as can be seen in Section 5 and it can be explicitly computed.*

Our last result is given by the following corollary.

Corollary 1.5. *With the same hypotheses as in the previous theorem, we have that*

$$\sum L^{(\mu)}(\tfrac{1}{2}, \chi_{\gamma m}) \sim \frac{(-2\log q)^{\mu}}{\mu+1}\left(1-\frac{1}{q}\right) q^M \left(\frac{M-2}{2}\right)^{\mu+1}, \tag{1.19}$$

as $M \to \infty$.

As an example of Theorem 1.3 we have the following formulas for the first and the second derivative at the central point.

$$\sum L^{(1)}(\tfrac{1}{2}, \chi_{\gamma m}) = (-1+q) q^{-1+\frac{M}{2}} (\log q)$$
$$\times \left(-\frac{\left(M - \sqrt{q} + M\sqrt{q}\right)\left(-\sqrt{q}\right)^M}{\left(1+\sqrt{q}\right)^2} - \frac{\sqrt{q}}{\left(1+\sqrt{q}\right)^2} - \frac{1}{4}(-2+M) M q^{M/2}\right).$$

$$\sum L^{(2)}(\tfrac{1}{2}, \chi_{\gamma m}) = \frac{1}{3}(-1+q) q^{-1+\frac{M}{2}} (\log q)^2$$
$$\times \left(\frac{1}{2} M \left(2 - 3M + M^2\right) q^{M/2}\right.$$
$$\left. + \frac{3\left(M^2\left(1+\sqrt{q}\right)^2\left(-\sqrt{q}\right)^M + 2M\left(1+\sqrt{q}\right)\left(-\sqrt{q}\right)^{-1+M} q + \left(-1+\left(-\sqrt{q}\right)^M\right)\left(-\sqrt{q}+q\right)\right)}{\left(1+\sqrt{q}\right)^3}\right).$$

2. Preparatory Results

It follows from equation (1.5) that the averages of $L(s, \chi_m)$ and its derivatives over polynomials m of a fixed degree mainly reduces to averaging the sums $S_k(\chi_m)$. To handle these particular character sums we will need the following definition and propositions.

Definition 2.1. *Let M and N be non-negative integers, and n a monic of degree N. Define $\Phi_n(M)$ to be the number of monic polynomials m of degree M such that $\gcd(n, m) = 1$. Define $\Phi(N, M)$ to be the number of pairs (n, m) of monic polynomials such that $\deg(n) = N$, $\deg(m) = M$, and $\gcd(n, m) = 1$.*

Note that

$$\sum_{\substack{n \text{ monic} \\ \deg(n)=N}} \Phi_n(M) = \Phi(N, M), \tag{2.1}$$

and that

$$\Phi(N, M) = \Phi(M, N). \tag{2.2}$$

The next result is taken from Hoffstein and Rosen [6] and for completeness, we repeat the proof here.

Proposition 2.2. *We have that*

(i)
$$\Phi(0, M) = q^M.$$

(ii) *If $M, N \geq 1$, then*
$$\Phi(N, M) = q^{M+N}\left(1 - \frac{1}{q}\right).$$

Proof. Part (i) is trivial. To prove (ii), note that

$$q^{M+N} = \sum_{d=0}^{\min(M,N)} q^d \Phi(M - d, N - d).$$

This follows by partitioning the set $\{(f, g) | f, g \text{ monic}, \ \deg(f) = M, \ \deg(g) = N\}$ in the obvious way and then a straightforward induction completes the proof. □

It is convenient to extend the definition of $\Phi(N, M)$ by defining $\Phi(k/2, M) = 0$ if k is odd. The next proposition will be used in the paper to handle the averages of $S_k(\chi_m)$ and is due to Hoffstein and Rosen [6, Proposition 1.3] and we repeat the proof here as it is an illuminating result.

Proposition 2.3. *Suppose $0 \leq k \leq M-1$. Then*

$$\sum_{\deg(m)=M} S_k(\chi_m) = \Phi(k/2, M). \tag{2.3}$$

Proof. We assume all sums are over monic polynomials. We can write

$$\sum_{\deg(m)=M} S_k(\chi_m) = \sum_{\deg(m)=M} \sum_{\deg(n)=k} \left(\frac{m}{n}\right) = \sum_{\deg(n)=k} \sum_{\deg(m)=M} \left(\frac{m}{n}\right).$$

If n is not a square of a polynomial, then we have

$$\sum_{\deg(m)=M} \left(\frac{m}{n}\right) = 0$$

since $(*/n)$ is a non-trivial character modulo n, and $M > k$. This follows from the fact that the L-function associated to χ_m is a finite sum.

If n is a square of a polynomial, say $n = n_1^2$ then we have

$$\sum_{\deg(m)=M} \left(\frac{m}{n}\right) = \sum_{\deg(m)=M} \left(\frac{m}{n_1}\right)^2 = \Phi_{n_1}(M).$$

Thus

$$\sum_{\deg(m)=M} S_k(\chi_m) = \sum_{\deg(n_1)=k/2} \Phi_{n_1}(M) = \Phi\left(\frac{k}{2}, M\right).$$

To handle the general case, let $\alpha \in \mathbb{F}_q^*$ and sum over all αm as m runs through the monics of degree M. The above calculation shows that the answer is again $\Phi(k/2, M)$. □

The last preliminary result we need is known as Faulhaber's formula [4, 9] and expresses the sum of the p-th powers of the first n positive integers

$$\sum_{k=1}^{n} k^p = 1^p + 2^p + 3^p + \cdots \tag{2.4}$$

as a $(p+1)$th-degree polynomial function of n, the coefficients involving Bernoulli numbers B_j

$$\sum_{k=1}^{n} k^p = \frac{1}{p+1} \sum_{j=0}^{p} (-1)^j \binom{p+1}{j} B_j n^{p+1-j}, \tag{2.5}$$

where $B_1 = -\frac{1}{2}$.

A variant of Faulhaber's formula [4, 9] gives that

$$\sum_{j=1}^{M-1} j^{\mu}(q^{\frac{1}{2}})^{j} = q^{-\frac{M}{2}}\Phi(\sqrt{q}, -\mu, M) + \mathrm{Li}_{-\mu}(\sqrt{q}), \tag{2.6}$$

and

$$\sum_{j=1}^{M-1} j^{\mu}(-1)^{j}(q^{\frac{1}{2}})^{j} = -(-\sqrt{q})^{M}\Phi(-\sqrt{q}, -\mu, M) + \mathrm{Li}_{-\mu}(-\sqrt{q}). \tag{2.7}$$

3. Proof of Theorem 1.1

Taking the μ-th derivative of equation (1.5) with respect to s we obtain

$$L^{(\mu)}(s, \chi_m) = (-1)^{\mu}(\log q)^{\mu}\sum_{d=0}^{M-1} S_d(\chi_m)d^{\mu}q^{-ds}. \tag{3.1}$$

The aim is to establish an exact formula for

$$\sum_{\deg(m)=M}(-1)^{\mu}(\log q)^{\mu}\sum_{d=0}^{M-1} S_d(\chi_m)d^{\mu}q^{-ds} \tag{3.2}$$

when $s = 1/2$ and $M > 0$ is an odd integer.

We have that

$$\sum_{\deg(m)=M} L^{(\mu)}(s, \chi_m) = (-1)^{\mu}(\log q)^{\mu}\sum_{d=0}^{M-1} d^{\mu}q^{-ds}S_d(\chi_m). \tag{3.3}$$

By invoking Proposition 2.3 and the fact that $\Phi(k/2, M) = 0$ if k is odd, we have that

$$\begin{aligned}\sum_{\deg(m)=M} L^{(\mu)}(s, \chi_m) &= (-1)^{\mu}(\log q)^{\mu}\sum_{d=0}^{M-1} d^{\mu}q^{-ds}\Phi(d/2, M)\\ &= (-1)^{\mu}(\log q)^{\mu}\Bigg(2^{\mu}q^{-2s}\Phi(1, M) + 4^{\mu}q^{-4s}\Phi(2, M)\\ &+ \cdots + (M-1)^{\mu}q^{-(M-1)s}\Phi\left(\frac{M-1}{2}, M\right)\Bigg)\end{aligned} \tag{3.4}$$

Using Proposition 2.2 and after a few arithmetic moves we obtain

$$\sum_{\deg(m)=M} L^{(\mu)}(s,\chi_m) = (-1)^\mu (\log q)^\mu \left(1-\frac{1}{q}\right) q^M$$
$$\times \left(2^\mu q^{1-2s} + 4^\mu (q^{1-2s})^2 + \cdots + (M-1)^\mu (q^{1-2s})^{\frac{M-1}{2}}\right). \tag{3.5}$$

For $s=\frac{1}{2}$ we have

$$\sum_{\deg(m)=M} L^{(\mu)}(\tfrac{1}{2},\chi_m) = (-1)^\mu (\log q)^\mu \left(1-\frac{1}{q}\right) q^M \left(2^\mu + 4^\mu + \cdots + (M-1)^\mu\right)$$
$$= (-1)^\mu (\log q)^\mu \left(1-\frac{1}{q}\right) q^M \sum_{j=1}^{\frac{M-1}{2}} (2j)^\mu. \tag{3.6}$$

Using Faulhaber's formula (2.5) we obtain

$$\sum_{\deg(m)=M} L^{(\mu)}(\tfrac{1}{2},\chi_m) = (-1)^\mu (\log q)^\mu \left(1-\frac{1}{q}\right) q^M 2^\mu$$
$$\times \frac{1}{\mu+1} \sum_{l=0}^{\mu} (-1)^l \binom{\mu+1}{l} B_l \left(\frac{M-1}{2}\right)^{\mu+1-l}. \tag{3.7}$$

And this proves the theorem.

□

4. Proof of Theorem 1.2

We assume now that $M>0$ is an even integer. The aim is to obtain an exact formula for

$$\sum_{\substack{\deg(m)=M \\ m \text{ non-square}}} L^{(\mu)}(s,\chi_m). \tag{4.1}$$

We have that,

$$\begin{aligned}
\sum_{\substack{\deg(m)=M \\ m \text{ non-square}}} L^{(\mu)}(s,\chi_m) &= \sum_{\deg(m)=M} L^{(\mu)}(s,\chi_m) - \sum_{\substack{\deg(m)=M \\ m \text{ a square}}} L^{(\mu)}(s,\chi_m) \\
&= (-1)^{\mu}(\log q)^{\mu} \sum_{d=0}^{M-1} d^{\mu} q^{-ds} \sum_{\deg(m)=M} S_d(\chi_m) \\
&\quad - (-1)^{\mu}(\log q)^{\mu} \sum_{d=0}^{M-1} d^{\mu} q^{-ds} \sum_{\substack{\deg(m)=M \\ m \text{ a square}}} S_d(\chi_m).
\end{aligned} \tag{4.2}$$

To proceed with the proof of the theorem we need an extra result. If $0 \leq k < M$, and $\deg(n) = k$,

$$\begin{aligned}
\sum_{\substack{\deg(m)=M \\ m \text{ a square}}} \chi_m(n) &= \sum_{\deg(m_1)=\frac{M}{2}} \left(\frac{m_1^2}{n}\right) \\
&= \textstyle\sum_{\deg(m_1)=\frac{M}{2}} \left(\frac{m_1}{n}\right)^2 \\
&= \Phi_n\left(\frac{M}{2}\right),
\end{aligned} \tag{4.3}$$

where the last equality follows because $(\frac{m_1}{n})^2 = 1$ if $(m_1, n) = 1$ and 0 otherwise.

By (??) it follows that

$$\sum_{\substack{\deg(m)=M \\ m \text{ a square}}} S_k(\chi_m) = \Phi\left(k, \frac{M}{2}\right). \tag{4.4}$$

Combining Proposition 2.3 and the equation above we obtain that

$$\begin{aligned}
\sum_{\substack{\deg(m)=M \\ m \text{ non-square}}} L^{(\mu)}(s,\chi_m) &= (-1)^{\mu}(\log q)^{\mu} \sum_{d=0}^{M-1} d^{\mu} q^{-ds} \Phi\left(\frac{d}{2}, M\right) \\
&\quad - (-1)^{\mu}(\log q)^{\mu} \sum_{d=0}^{M-1} d^{\mu} q^{-ds} \Phi\left(d, \frac{M}{2}\right).
\end{aligned} \tag{4.5}$$

We seek to better understand the two sums over d in the previous equation. We compute them separately. Using that $\Phi(N,M) = q^{M+N}(1-1/q)$, $\Phi(N/2, M) = 0$ if N is odd we obtain that the first sum over d is

$$
\begin{aligned}
&(-1)^{\mu}(\log q)^{\mu}\sum_{d=0}^{M-1} d^{\mu} q^{-ds}\Phi\left(\frac{d}{2},M\right)\\
&= (-1)^{\mu}(\log q)^{\mu}\Bigg(2^{\mu}q^{-2s}\Phi(1,M)+4^{\mu}q^{-4s}\Phi(2,M)+6^{\mu}q^{-6s}\Phi(3,M)\\
&+\cdots+(M-2)^{\mu}q^{-(M-2)s}\Phi\left(\frac{M-2}{2},M\right)\Bigg) \qquad (4.6)\\
&= (-1)^{\mu}(\log q)^{\mu}\left(1-\frac{1}{q}\right)q^{M}\Bigg(2^{\mu}q^{1-2s}+4^{\mu}(q^{1-2s})^{2}+6^{\mu}(q^{1-2s})^{3}\\
&+\cdots+(M-2)^{\mu}(q^{1-2s})^{\frac{M-2}{2}}\Bigg).
\end{aligned}
$$

For $s=\frac{1}{2}$ we have that

$$
(-1)^{\mu}(\log q)^{\mu}\sum_{d=0}^{M-1} d^{\mu} q^{-d/2}\Phi\left(\frac{d}{2},M\right) = (-1)^{\mu}(\log q)^{\mu}\left(1-\frac{1}{q}\right)q^{M}(2)^{\mu}\sum_{j=1}^{\frac{M-2}{2}} j^{\mu}, \qquad (4.7)
$$

and using Faulhaber's formula (2.5) we get

$$
\begin{aligned}
&(-1)^{\mu}(\log q)^{\mu}\sum_{d=0}^{M-1} d^{\mu} q^{-d/2}\Phi\left(\frac{d}{2},M\right)\\
&= (-1)^{\mu}(\log q)^{\mu}(2)^{\mu}\left(1-\frac{1}{q}\right)q^{M}\frac{1}{\mu+1}\sum_{l=0}^{\mu}(-1)^{l}\binom{\mu+1}{l}B_{l}\left(\frac{M-2}{2}\right)^{\mu+1-l}. \qquad (4.8)
\end{aligned}
$$

For the second sum over d in (4.5), after using $\Phi(M,N)=\Phi(N,M)$ and Proposition 2.2, we obtain

$$
\begin{aligned}
&-(-1)^{\mu}(\log q)^{\mu}\sum_{d=0}^{M-1} d^{\mu}q^{-ds}\Phi\left(d,\frac{M}{2}\right)\\
&=-(-1)^{\mu}(\log q)^{\mu}\Bigg(q^{-s}\Phi\left(\frac{M}{2},1\right)+2^{\mu}q^{-2s}\Phi\left(\frac{M}{2},2\right)\\
&+\cdots+(M-1)^{\mu}q^{-(M-1)s}\Phi\left(\frac{M}{2},M-1\right)\Bigg)\\
&=-(-1)^{\mu}(\log q)^{\mu}\left(1-\frac{1}{q}\right)q^{M/2}\left(q^{1-s}+2^{\mu}(q^{1-s})^{2}+\cdots+(M-1)^{\mu}(q^{1-s})^{M-1}\right).
\end{aligned}
\tag{4.9}
$$

For $s=\frac{1}{2}$ and using a variant of Faulhaber's formula (2.6) we obtain that

$$
\begin{aligned}
&-(-1)^{\mu}(\log q)^{\mu}\sum_{d=0}^{M-1} d^{\mu}q^{-d/2}\Phi\left(d,\frac{M}{2}\right)\\
&=-(-1)^{\mu}(\log q)^{\mu}\left(1-\frac{1}{q}\right)q^{\frac{M}{2}}\sum_{j=1}^{M-1} j^{\mu}(q^{\frac{1}{2}})^{j}\\
&=-(-1)^{\mu}(\log q)^{\mu}\left(1-\frac{1}{q}\right)q^{\frac{M}{2}}\left(q^{-\frac{M}{2}}\Phi(\sqrt{q},-\mu,M)+\mathrm{Li}_{-\mu}(\sqrt{q})\right).
\end{aligned}
\tag{4.10}
$$

Combining equations (4.8) and (4.10) we establish the desired result.

□

5. Proof of Theorem 1.3

We assume now that $M>0$ is an even integer and that $\gamma\in\mathbb{F}_q^{*}$ is a non-square. The aim is to obtain an exact formula for

$$
\sum_{\substack{\deg(m)=M\\ m \text{ non-square}}} L^{(\mu)}(s,\chi_{\gamma m})
\tag{5.1}
$$

when $s=\frac{1}{2}$.

We have that

$$\sum_{\substack{\deg(m)=M\\ m \text{ non-square}}} L^{(\mu)}(s,\chi_{\gamma m}) = \sum_{\deg(m)=M} L^{(\mu)}(s,\chi_{\gamma m}) - \sum_{\substack{\deg(m)=M\\ m \text{ a square}}} L^{(\mu)}(s,\chi_{\gamma m})$$

$$= (-1)^{\mu}(\log q)^{\mu} \sum_{d=0}^{M-1} d^{\mu} q^{-ds} \sum_{\deg(m)=M} S_d(\chi_{\gamma m}) \tag{5.2}$$

$$- (-1)^{\mu}(\log q)^{\mu} \sum_{d=0}^{M-1} d^{\mu} q^{-ds} \sum_{\substack{\deg(m)=M\\ m \text{ square}}} S_d(\chi_{\gamma m}).$$

Using the definition of $S_d(\chi_m)$ we get

$$\sum_{\substack{\deg(m)=M\\ m \text{ non-square}}} L^{(\mu)}(s,\chi_{\gamma m}) = (-1)^{\mu}(\log q)^{\mu} \sum_{d=0}^{M-1} d^{\mu} q^{-ds} \sum_{\deg(m)=M} \sum_{\deg(n)=d} \chi_{\gamma m}(n)$$

$$- (-1)^{\mu}(\log q)^{\mu} \sum_{d=0}^{M-1} d^{\mu} q^{-ds} \sum_{\substack{\deg(m)=M\\ m \text{ square}}} \sum_{\deg(n)=d} \chi_{\gamma m}(n). \tag{5.3}$$

To proceed with the proof of the theorem we need an extra result. Suppose n is monic of degree $k < M$. Then

$$\sum_{\substack{\deg(m)=M\\ m \text{ a square}}} \chi_{\gamma m}(n) = \sum_{\deg(m_1)=\frac{M}{2}} \left(\frac{\gamma m_1^2}{n}\right) = \left(\frac{\gamma}{n}\right) \Phi_n \left(\frac{M}{2}\right), \tag{5.4}$$

where the last equality follows from the multiplicativity of the Jacobi symbol and the fact that $(\frac{m_1}{n})^2 = 1$ if $(m_1, n) = 1$ and 0 otherwise. Let $f = \alpha_0 T^d + \alpha_1 T^{d-1} + \cdots + \alpha_d$ be a polynomial in A. We define $\operatorname{sgn}(f) = 1$ if $\alpha_0 \in (\mathbb{F}_q^*)^2$ and $\operatorname{sgn}(a) = -1$ if $\alpha_0 \notin (\mathbb{F}_q^*)^2$. It is easy to show that $\left(\frac{\gamma}{n}\right) = \operatorname{sgn}(\gamma)^{\deg(n)} = (-1)^k$. Applying (**??**), we then get that

$$\sum_{\substack{\deg(m)=M\\ m \text{ a square}}} S_k(\chi_m) = (-1)^k \Phi\left(k, \frac{M}{2}\right). \tag{5.5}$$

Using equations (5.3) and (5.4) we obtain

$$\sum_{\substack{\deg(m)=M \\ m \text{ non-square}}} L^{(\mu)}(s,\chi_{\gamma m}) = (-1)^\mu (\log q)^\mu \sum_{d=0}^{M-1} d^\mu q^{-ds} \sum_{\deg(m)=M} \sum_{\deg(n)=d} \chi_{\gamma m}(n)$$
$$- (-1)^\mu (\log q)^\mu \sum_{d=0}^{M-1} d^\mu q^{-ds} \sum_{\deg(n)=d} \left(\frac{\gamma}{n}\right) \Phi_n \left(\frac{M}{2}\right). \tag{5.6}$$

Invoking Proposition (2.2), Proposition (2.3), equation (5.5) and the definition of $\mathrm{sgn}(\gamma)$ we obtain

$$\sum_{\substack{\deg(m)=M \\ m \text{ non-square}}} L^{(\mu)}(s,\chi_{\gamma m}) = (-1)^\mu (\log q)^\mu \sum_{d=0}^{M-1} d^\mu q^{-ds} \Phi\left(\frac{d}{2}, M\right)$$
$$- (-1)^\mu (\log q)^\mu \sum_{d=0}^{M-1} d^\mu q^{-ds} (-1)^d \Phi\left(d, \frac{M}{2}\right). \tag{5.7}$$

Again, we need to handle the two sums over d in the previous equation. We compute them separately. Using that $\Phi(N,M) = q^{M+N}(1-1/q)$ and $\Phi(N/2, M) = 0$ if N is odd we obtain that the first sum over d is

$$(-1)^\mu (\log q)^\mu \sum_{d=0}^{M-1} d^\mu q^{-ds} \Phi\left(\frac{d}{2}, M\right)$$
$$= (-1)^\mu (\log q)^\mu \Bigg(2^\mu q^{-2s} \Phi(1,M) + 4^\mu q^{-4s} \Phi(2,M)$$
$$+ \cdots + (M-2)^\mu q^{-(M-2)s} \Phi\left(\frac{M-2}{2}, M\right)\Bigg)$$
$$= (-1)^\mu (\log q)^\mu \left(1 - \frac{1}{q}\right) q^M \left(2^\mu q^{1-2s} + 4^\mu (q^{1-2s})^2 + \cdots + (M-2)^\mu (q^{1-2s})^{\frac{M-2}{2}}\right). \tag{5.8}$$

For $s = \frac{1}{2}$ and using Faulhaber's formula (2.5) we obtain

$$(-1)^{\mu}(\log q)^{\mu}\sum_{d=0}^{M-1} d^{\mu}q^{-d/2}\Phi\left(\frac{d}{2},M\right)$$
$$=(-1)^{\mu}(\log q)^{\mu}(2)^{\mu}\left(1-\frac{1}{q}\right)q^{M}\frac{1}{\mu+1}\sum_{l=0}^{\mu}(-1)^{l}\binom{\mu+1}{l}B_{l}\left(\frac{M-2}{2}\right)^{\mu+1-l}.$$
(5.9)

For the second sum over d in (5.7), after using that $\Phi(N,M)=\Phi(M,N)$ and Proposition 2.2, we obtain

$$-(-1)^{\mu}(\log q)^{\mu}\sum_{d=0}^{M-1} d^{\mu}q^{-ds}(-1)^{d}\Phi\left(d,\frac{M}{2}\right)$$
$$=-(-1)^{\mu}(\log q)^{\mu}\left(1-\frac{1}{q}\right)q^{\frac{M}{2}}\Bigg(1^{\mu}(-1)q^{1-s}+2^{\mu}(-1)^{2}(q^{1-2s})^{2}$$
$$+\cdots+(M-1)^{\mu}(-1)^{M-1}(q^{1-2s})^{M-1}\Bigg).$$
(5.10)

For $s=\frac{1}{2}$ and using a variant of Faulhaber's formula (2.7) we obtain that

$$-(-1)^{\mu}(\log q)^{\mu}\sum_{d=0}^{M-1} d^{\mu}q^{-d/2}(-1)^{d}\Phi\left(d,\frac{M}{2}\right)$$
$$=-(-1)^{\mu}(\log q)^{\mu}\left(1-\frac{1}{q}\right)q^{\frac{M}{2}}\sum_{j=1}^{M-1} j^{\mu}(-1)^{j}(q^{\frac{1}{2}})^{j}$$
$$=-(-1)^{\mu}(\log q)^{\mu}\left(1-\frac{1}{q}\right)q^{\frac{M}{2}}\left(-(-\sqrt{q})^{M}\Phi(-\sqrt{q},-\mu,M)+\mathrm{Li}_{-\mu}(-\sqrt{q})\right).$$
(5.11)

Combining equations (5.9) and (5.11) establishes the theorem.

□

Acknowledgement. We would like to express our gratitude to an anonymous referee whose comments improved the presentation of the paper.

References

[1] Andrade, J.C., Rajagopal S.: Mean values of derivatives of L-functions in function fields: I, J. Math. Anal. Appl., Vol. 443, Issue 1, 526-541 (2016).

[2] Conrey, B.: The Fourth Moment of Derivatives of the Riemann zeta-function, Q.J.Math., **39**, 21–36 (1988)

[3] Conrey, B., Rubinstein, M., Snaith, N.: Moments of the Derivative of Characteristic Polynomials with an Application to the Riemann Zeta Function, Commun. Math. Phys., **267**, 611-629 (2006).

[4] Conway, John H., Guy, Richard: The Book of Numbers. Springer-Verlag, New York (1996).

[5] Gonek, S.: Mean values of the Riemann zeta-function and its derivatives, Invent. Math., **75**, 123–141 (1984).

[6] Hoffstein, J., Rosen, M.: Average values of L-series in function fields, J. Reine Angew. Math., **426**, 117-150 (1992).

[7] Hughes C., Keating J.P. and O'Connell, N.: Random matrix theory and the derivative of the Riemann zeta function, Proc. R. Soc. Lond. Proc. A., **456**, 2611–2627 (2000).

[8] Ingham, A.E.: Mean-value theorems in the theory of the Riemann zeta-function, Proc. Lond. Math. Soc., **27**, 273–300 (1926).

[9] Knuth, Donald E.: Johann Faulhaber and sums of powers, Math. Comp., **61** (203), 277-294 (1993).

[10] Laurinčikas, A. and Steuding, J.: A Note on Moments of $\zeta^{'}(1/2 + i\gamma)$, Publications de l'Institut Mathematique, Volume 76, Issue 90, 57–63 (2004).

[11] Milinovich, M.: Upper bounds for moments of $\zeta^{'}(\rho)$, Bull. London Math. Soc., **42**, no. 1, 28-44 (2010).

[12] Milinovich, M. and Ng, N.: Lower bounds for moments of $\zeta^{'}(\rho)$, Int. Math. Res. Notices, no. 12, 3190-3216 (2014).

[13] Rosen, M.: Number Theory in Function Fields. Graduate Texts in Mathematics vol. 210. Springer-Verlag, New York (2002).

Department of Mathematics, University of Exeter, Exeter, EX4 4QF, United Kingdom

E-mail address: `j.c.andrade@exeter.ac.uk`