ECM3704 NUMBER THEORY **EXERCISE SHEET 4**

This sheet does **not** count towards assessment.

1. Find all solutions in integers to the following Diophantine equations (or show that there are none):

(i) $3x + 5y = 7$;
(ii) $4x - 6y = 3$;
(iii) $4x - 6y = 10$;
(iv) $x^2 - 7y = 4$;
(v) $x^2 + 4y^2 = 25$;
(vi) $x^2 + 1 = 7y^2 + 14x^3y^4$;
(vii) $x^2 - y^2 = 15$.

2. For each of the following numbers n, determine whether n can be written as the sum of two squares and, if it can, write n in that form.
(i) 34; (ii) 53; (iii) 67; (iv) 73; (v) 99; (vi) 229;
(vii) $3185 = 5 \cdot 7^2 \cdot 13$; (viii) $5075 = 5^2 \cdot 7 \cdot 29$; (ix) $39690 = 2 \cdot 3^4 \cdot 5 \cdot 7^2$.

3. If $n = a^2 + b^2$, we will regard any of the 8 expressions $(\pm a)^2 + (\pm b)^2$ and $(\pm b)^2 + (\pm a)^2$ for n as equivalent to $a^2 + b^2$. Find two inequivalent expressions for $377 = 13 \cdot 29$ as the sum of two squares. Do the same for $3869 = 53 \cdot 73$. Find four inequivalent representations of $112201 = 29 \cdot 53 \cdot 73$.

4. Show that a positive integer n can be expressed as the difference of two integer squares if and only if $n \not\equiv 2 \pmod 4$.

5. Write 29 and 43 as sums of 4 squares, and hence write $1247 = 29 \cdot 43$ in that form.

6. Show that if $n \geq 170$ then n can be written as the sum of 5 *positive* squares. [Hint: Write $m = n - 169$ as a sum of 4 integer squares and use the fact that $169 = 13^2 = 5^2 + 12^2 = 3^2 + 4^2 + 12^2 = 1^2 + 2^2 + 8^2 + 10^2$.]

7. Show that in every Pythagorean triple (x, y, z), at least one of x, y is divisible by 3, and at least one of x, y, z is divisible by 5.

8. Prove or disprove the following statement: If (x, y, z) is a Pythagorean triple (not necessarily primitive) then, after swapping x and y if necessary, there are integers r, s such that $x = r^2 - s^2$, $y = 2rs$, $z = r^2 + s^2$.