# EXERCISE SHEET 4 – SOLUTIONS

**This sheet does not count for assessment**

1. (i) $3x + 5y = 7$: first solve $3x \equiv 7 \pmod 5$. This has solution $x \equiv -1 \pmod 5$. Write this as $x = 5a - 1$ and substitute into original equation to get $3(5a - 1) + 5y = 7$. This simplifies to $y = 2 - 3a$. Thus the general solution is $x = 5a - 1$, $y = 2 - 3a$ for arbitrary $a \in \mathbb{Z}$.

(ii) $4x - 6y = 3$ has **no integer solutions** since the left-hand side is even for any choice of $x$, $y \in \mathbb{Z}$ while the right-hand side is odd.

(iii) $4x-6y = 10$ is equivalent to $2x-3y = 5$. Solving $2x \equiv 5 \pmod 3$ we get $x \equiv 1 \pmod 3$, so that $x = 3a+1$. Substituting, we have $2(3a+1)-3y = 5$, so that $y = 2a - 1$. Hence the general solution is $x = 3a + 1$, $y = 2a - 1$ for arbitrary $a \in \mathbb{Z}$.

(iv) $x^2 - 7y = 4$: Solving $x^2 \equiv 4 \pmod 7$ we get $x \equiv \pm 2 \pmod 7$, so that $x = \pm(2 + 7a)$. Substituting, we have $(2 + 7a)^2 - 7y = 4$. Expanding and simplifying, we find that $y = 4a + 7a^2$. Hence the general solution is $x = \pm(2 + 7a)$, $y = 4a + 7a^2$ for arbitrary $a \in \mathbb{Z}$.

(v) $x^2 + 4y^2 = 25$. For any integer solution we must have $y^2 \leq 25/4$, so by exhaustive search the solutions are $(x, y) = (\pm 5, 0)$ or $(\pm 3, \pm 2)$.

(vi) $x^2 + 1 = 7y^2 + 14x^3y^4$: for any solution of this we would have $x^2 + 1 \equiv 0 \pmod 7$, which is impossible. Hence there are **no integer solutions**.

(vii) $x^2 - y^2 = 15$. Clearly if $(x, y)$ is a solution then so is any of $(\pm x, \pm y)$, so it suffices to find solutions with $x$, $y \geq 0$. Now $x^2 - y^2 = (x + y)(x - y)$, so look at ways of writing 15 as a product of two factors $x + y$ and $x - y$ with $x + y \geq x - y > 0$. The only possibilities are $x + y = 15$, $x - y = 1$ and $x + y = 5$, $x - y = 3$. This gives the solutions $(x, y) = (8, 7)$ or $(4, 1)$. Hence the full list of solutions is $(x, y) = (\pm 8, \pm 7)$ or $(\pm 4 \pm 1)$.

2. (i) $34 = (1^2 + 1^2)(1^2 + 4^2) = 3^2 + 5^2$.

(ii) $53 = 2^2 + 7^2$.

(iii) 67 cannot be written as the sum of two squares, since 67 is prime and $67 \equiv 3 \pmod 4$.

(iv) $73 = 3^2 + 8^2$.

(v) 99 cannot be written as the sum of two squares, since $v_{11}(99) = 1$.

(vi) $229 = 2^2 + 15^2$.

(vii)

$$3185 = 5 \cdot 7^2 \cdot 13 = 7^2(1^2 + 2^2)(2^2 + 3^2) = 7^2(1^2 + 8^2) = 7^2 + 56^2.$$

(Alternatively, $3185 = 28^2 + 49^2$.)

(viii) $5075 = 5^2 \cdot 7 \cdot 29$ cannot be written as the sum of two squares since $v_7(5075) = 1$.

(ix) $39690 = 2 \cdot 3^4 \cdot 5 \cdot 7^2 = 63^2(1^2 + 3^2) = 63^2 + 189^2$.

3. We use the formula $(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2$, which we can obtain by expanding $|z|^2|w|^2 = |zw|^2$ for $z = a + ib$, $w = c + id$:

$377 = 13 \cdot 29 = (2^2 + 3^2)(2^2 + 5^2) = 11^2 + 16^2$,
and also $377 = (2^2 + 3^2)(5^2 + 2^2) = 4^2 + 19^2$.

$3869 = 53 \cdot 73 = (2^2 + 7^2)(3^2 + 8^2) = 50^2 + 37^2$,
and also $3869 = (2^2 + 7^2)(8^2 + 3^2) = 5^2 + 62^2$.

$112201 = 29 \cdot 53 \cdot 73 = (2^2 + 5^2)(50^2 + 37^2) = (2^2 + 5^2)(5^2 + 62^2)$, so

$$112201 = 85^2 + 324^2 = 176^2 + 285^2 = 300^2 + 149^2 = 99^2 + 320^2.$$

4. We can write the equation $n = x^2 - y^2$ as $n = (x + y)(x - y)$.

If $n$ is odd we can put $x + y = n$, $x - y = 1$. Then $x = (n + 1)/2$ and $y = (n - 1)/2$. We then have $x, y \in \mathbb{Z}$ and $x^2 - y^2 = n$.

If $n = 4k$ with $k \in \mathbb{Z}$, put $x = k+1$, $y = k-1$. We then have $x^2 - y^2 = 4k = n$.

We have now shown that we can solve $x^2 - y^2 = n$ whenever $n \not\equiv 2 \pmod 4$.

Conversely, if $n = x^2 - y^2$ then we have $x^2, y^2 \equiv 0$ or $1 \pmod 4$, so $n \equiv 0 - 0, 0 - 1, 1 - 0$ or $1 - 1 \pmod 4$. Thus we cannot have $n \equiv 2 \pmod 4$.

5. By inspection we have $29 = 5^2 + 2^2 + 0^2 + 0^2$ and $43 = 5^2 + 3^2 + 3^2 + 0^2$. Now from lectures we have

$$(a_1^2 + a_2^2 + a_3^2 + a_4^2)(b_1^2 + b_2^2 + b_3^2 + b_4^2) = (c_1^2 + c_2^2 + c_3^2 + c_4^2)$$

where

$$\begin{aligned} c_1 &= a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4 \\ c_2 &= a_1b_2 - a_2b_1 + a_3b_4 - a_4b_3 \\ c_3 &= a_1b_3 - a_2b_4 - a_3b_1 + a_4b_2 \\ c_4 &= a_1b_4 + a_2b_3 - a_3b_2 - a_4b_1. \end{aligned}$$

Taking $a_1 = 5$, $a_2 = 2$, $a_3 = a_4 = 0$ and $b_1 = 5$, $b_2 = b_3 = 3$, $b_4 = 0$, we get $c_1 = 25 + 6 = 31$, $c_2 = 15 - 10 = 5$, $c_3 = 15$, $c_4 = 6$. Hence

$$1247 = 31^2 + 5^2 + 15^2 + 6^2.$$

(There are other solutions.)

6. Given $n \geq 170$, write $n - 169 \in \mathbb{N}$ as a sum of four integer squares:

$$n - 169 = a^2 + b^2 + c^2 + d^2,$$

where, without loss of generality, $a \geq b \geq c \geq d \geq 0$. (This is always possible by Lagrange's Four Squares Theorem.)
If $d > 0$ then $n = a^2 + b^2 + c^2 + d^2 + 13^2$.
If $c > 0$ and $d = 0$ then $n = a^2 + b^2 + c^2 + 12^2 + 5^2$.
If $b > 0$ and $c = d = 0$ then $n = a^2 + b^2 + 12^2 + 4^2 + 3^2$.
If $a > 0$ and $b = c = d = 0$ then $n = a^2 + 10^2 + 8^2 + 2^2 + 1^2$.
Clearly we cannot have $a = b = c = d = 0$.
Hence in all cases we have expressed $n$ as a sum of 5 *positive* integer squares.

7. In either case, it suffices to consider only primitive Pythagorean triples.

Every primitive Pythagorean triple $(x, y, z)$ has the form

$$x = r^2 - s^2, \qquad y = 2rs, \qquad z = r^2 + s^2$$

(after swapping $x$, $y$ if necessary) for some $r$, $s \in \mathbb{N}$.

If $3 \nmid y$ then $3 \nmid r$, $3 \nmid s$, so we have $r$, $s \equiv \pm 1 \pmod 3$. Hence $r^2 \equiv s^2 \equiv 1 \pmod 3$, so $3 \mid x$. Hence either $x$ or $y$ is divisible by 3.

If $5 \nmid y$ then $5 \nmid r$, $5 \nmid s$, so we have $r$, $s \equiv \pm 1$ or $\pm 2 \pmod 5$. Hence $r^2, s^2 \equiv \pm 1 \pmod 5$. Thus either $r^2 \equiv s^2 \pmod 5$, so $x = r^2 - s^2$ is divisible by 5, or $r^2 \equiv -s^2 \pmod 5$, so $z = r^2 + s^2$ is divisible by 5. Hence at least one of $x$, $y$, $z$ is divisible by 5.

8. The statement is false: e.g. for the Pythagorean triple (9,12,15) we would need

$$9 = r^2 - s^2, \qquad 12 = 2rs \qquad 15 = r^2 + s^2.$$

Then $2s^2 = 15 - 9 = 6$, so that $s = \pm\sqrt{3} \notin \mathbb{Z}$.

[What we proved in lectures was that every *primitive* Pythagorean triple can be obtained by these formulae.]