ECM3704 - NUMBER THEORY 2016-17

EXERCISE SHEET 2 (ASSESSED)

Please hand in your solutions to the starred questions via BART by 12 noon on Monday 7th November 2016.

10 marks out of 100 will be for presentation (reasoning clearly expressed, correct use of notation, etc.) Please consult the *Guide to Basic Study Skills* available on ELE.

1*. Prove that if p is a prime and $a^2 \equiv b^2 \pmod{p}$, then $p \mid (a+b)$ or $p \mid (a-b)$. **[10]**

Total for question: [10]

2.

(i) Prove that if $d \mid a$ and $d \mid b$ and $d > 0$, then

$$\left(\frac{a}{d}, \frac{b}{d}\right) = \frac{1}{d}(a, b).$$

If $(a, b) = g$, then

$$\left(\frac{a}{g}, \frac{b}{g}\right) = 1.$$

(ii) Prove that $ax \equiv ay \pmod{m}$ if and only if $x \equiv y \bmod \left(\frac{m}{(a,m)}\right)$.

3. Prove that there are infinitely many primes p with $p \equiv 2 \pmod{3}$.

4. Solve the following congruences (giving the most general solution), or show that no solution exists:

(i)* $3x \equiv 10 \pmod{13}$ **[4]**

(ii) $12x \equiv 20 \pmod{38}$

(iii)* $20x \equiv 4 \pmod{30}$ **[2]**

(iv) $15x \equiv 43 \pmod{99}$

(v)* $353x \equiv 254 \pmod{400}$ **[4]**

Total for question: [10]

5*. Find all integer solutions (if any exist) to each of the following equations:

(i) $13x + 31y = 2$ **[5]**

(ii) $12x + 28y = 16$ **[5]**

Total for question: [10]

6*.

(i) Evaluate $\phi(m)$ for $m = 1, 2, 3, \ldots, 12$. **[3]**

(ii) Find the remainder when 245^{1040} is divided by 18. **[5]**

(iii) Find the smallest positive integer n so that $\phi(x) = n$ has no solution; exactly two solutions; exactly three solutions. **[12]**

Total for question: [20]

7*. Solve each of the following systems of simultaneous congruences or show that the systems are not solvable.

(i) $x \equiv 1 \pmod 3$, $x \equiv 2 \pmod 4$, and $x \equiv 3 \pmod 5$ **[5]**

(ii) $x \equiv 2 \pmod 5$ and $x \equiv 3 \pmod 7$ **[2]**

(iii) $x \equiv 2 \pmod 4$ and $x \equiv 3 \pmod 6$ **[3]**

Total for question: [10]

8*. Show that the product of three consecutive integers is divisible by 504 if the middle one is a cube. **[20]**

Total for question: [20]

9*. Solve $x^2 + x + 47 \equiv 0 \pmod{7^3}$. **[10]**

Total for question: [10]