대수구조 및 응용연구센터
Algebraic Structure and its Applications Research Center
305-701 대전광역시 유성구 과학로 335 (구성동 373-1)
Tel. (042) 869-8111
Fax. (042) 869-8114

KAIST
한국과학기술원
Korea Advanced Institute of Science and Technology

Title: Arthur's Multiplicity Formula for automorphic representations of certain inner forms of special orthogonal and symplectic groups.

by: Olivier Taibi (Imperial College) 29/10/2015

Theorem: F number field. G red. group$/F$ either $SO(V,q)$ or an inner form of a symplectic group. Assume $\exists$ set $S \neq \emptyset$ of real places of F s.t. :
(i) $\forall v \in S$, $G_{F_v} = G \times F_v$ has discrete series repn.
(ii) $\forall v \notin S$, G_{F_v} is quasi-split.

(say that (G,S) has property $*$)

Choose infinitesimal char. $(\nu_v)_{v|\infty}$. Let $\hat{\nu}_v$ be the $\mathrm{Aut}(G_{F_v})$-orbit of ν_v (singleton if G is SO_{2n+1} or inner form of Sp, 2 elts if $G = SO_{2n}$).

$$\mathcal{A}^2(G,\tilde{\nu}) \simeq \bigoplus_{\psi \in \tilde{\Psi}_{disc}(G,\tilde{\nu})} \Bigg(\bigoplus_{\substack{\pi \in \Pi_\psi \text{ s.t.} \\ \langle \cdot, \pi \rangle = \varepsilon_\psi}} \pi \Bigg)^{\oplus m_\psi}$$

$\mathcal{A}(G) =$ space of aut. forms for $G \subset \{ \text{functions } G(F) \backslash G(\mathbb{A}_F) \to \mathbb{C} \}$

$\Psi_{disc}(G)$ should be the set of $\hat{G}$-conj. classes of Arthur-Langlands parameters

$$L_F \times SL_2(\mathbb{C}) \to {}^L G$$

should be related to $\mathrm{Gal}_F := \mathrm{Gal}(\bar{F}/F)$

(compact top. group $\to L_F \to W_F$)

335 Gwahangno (373-1 Guseong-dong), Yuseong-gu, Daejeon 305-701, KOREA (Tel) +82-42-869-8111 (Fax) +82-42-869-8114

Arthur used estabilisation of the twisted TF for $GL_{N/F}$, $g \mapsto {}^t g^{-1}$
& ——— ordinary ——— SO and Sp.

Special case of property $*$: (i) $G(\mathbb{R} \otimes_{\mathbb{Q}} F)$ compact.
(ii) $\forall$ finite place v, G_{F_v} qs.

Example: 16-dim^l even unimodular lattices: (L, q) is even unimodular if L is a free finite rk $\mathbb{Z}$-module, q $\mathbb{Z}$-valued quadratic form, and $B_q(\cdot,\cdot)$ is non-deg. over $\mathbb{Z}$, $q > 0$.

X_{16} = set of iso. classes of such objects.

$$\{\underbrace{E_8 \oplus E_8}, E_{16}\} = X_{16}$$

L defines a red. group $G = SO(L)/\mathbb{Z}$. $G_{\mathbb{Z}_p}$ is the Chevalley gp. SO_{16}.

$$X_{16} \simeq G(\mathbb{Q}) \backslash G(\mathbb{A}_f) / G(\hat{\mathbb{Z}}).$$

Fact: $\left(L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))^{G(\hat{\mathbb{Z}})} \otimes W^*\right)^{G(\mathbb{R})} \approx \{ f : G(\mathbb{A}_f)/G(\hat{\mathbb{Z}}) \to W^*\}$

$\cancel{(W^*)^{SO(L_1)} \oplus (W^*)^{SO(L_2)}}$

$\approx (W^*)^{SO(L_1)} \otimes (W^*)^{SO(L_2)}$.

Thm $\Longrightarrow$ can compute the Hecke operators on $\{ f : X_{16} \to \mathbb{C}\}$

2 auto. rep^{ns} related to the Galois rep^{ns}

1) $(\underbrace{\chi^{-7} \oplus \chi^{-6} \oplus \dots \oplus \chi^{7}}_{l\text{-adic avatar of the 15-dim}^l \text{ irr. rep}^n \text{ of } SL_2}) \oplus 1$ where $\chi : \mathrm{Gal}_F \to \mathbb{Q}_l^*$ is the cyclotomic character.

2) $\underbrace{\rho_\Delta}_{\mathrm{Gal}_{\mathbb{Q}} \to GL_2(\mathbb{Q}_l)} \otimes (\chi^{7} \oplus \dots \oplus \chi^{4}) \oplus (\chi^{-3} \oplus \dots \oplus \chi^{3}) \oplus 1$

, $\Delta \in S_{12}(SL_2(\mathbb{Z}))$.

대수구조 및 응용연구센터
Algebraic Structure and its Applications Research Center
305-701 대전광역시 유성구 과학로 335 (구성동 373-1)
Tel. (042) 869-8111
Fax. (042) 869-8114

KAIST
한국과학기술원
Korea Advanced Institute of Science and Technology

Quasi-split red. gps.

1) $G^* = SO_{2n+1} = SO(V, q)$ $\qquad V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\oplus n} \oplus (1)$

$\hat{G} = Sp_{2n}(\mathbb{C})$

2) $G^* = SO_{2n}^{\alpha}$, $\alpha \in F^\times / F^{\times 2}$ $\qquad \alpha = 1 :$ $q \simeq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\oplus n}$

$= SO(V, q)$, $\hat{G} = SO_{2n}(\mathbb{C})$, $\alpha \neq 1 :$ $q \simeq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\oplus (m-1)} \oplus \begin{pmatrix} 1 & 0 \\ 0 & -\alpha \end{pmatrix}$

3.) $G^* = Sp_{2n} = Sp(V, a)$, V - $2n$-dim$^{\text{d}}$, $a(\cdot,\cdot)$ non alt. form

$\hat{G} = SO_{2n+1}(\mathbb{C})$

$$
{}^L G = \begin{cases} \hat{G} \times Gal_F & \text{if } G^* = Sp_{2n} \text{ or } SO_{2n+1} \text{ or } SO_{2n}^1 \\ \hat{G} \rtimes Gal_F & \end{cases}
$$

$$\downarrow$$

$$\hat{G} \rtimes Gal(F(\sqrt{\alpha})/F) \subset O_{2n}(\mathbb{C})$$

inner form G of G^* : $H^1(F, G^*_{ad}) \longleftarrow H^1(F, G^*)$

$G^*_{ad}(F)$ acts on $G^*(F) \rightsquigarrow$ action on ${}^L G$ is trivial.

$$H^1(u_{\mathbb{R}} \to \xi_{\mathbb{R}}, Z \to Sp_{2n}) \twoheadrightarrow H^1(\mathbb{R}, PGSp_{2n})$$

$$\underset{\substack{\text{is} \\ \hat{\mathbb{Z}}}}{u_{\mathbb{R}}} \hookrightarrow \xi_{\mathbb{R}} \twoheadrightarrow Gal(\mathbb{C}/\mathbb{R}) \qquad \text{Kaletha.}$$

elliptic endoscopic groups:

e.g. $G^* = Sp_{2n}$

$\hat{G} = SO_{2n+1}(\mathbb{C})$

$H = Sp_{2a} \times SO_{2b}^{\alpha}$

$\hat{H} = SO_{2a+1}(\mathbb{C}) \times SO_{2b}(\mathbb{C})$

$a, b \geq 0$

$a+b=n$

$\alpha = 1$ if $b=0$

Arthur's substitute for global parameters:

π auto. cusp - self-dual rep$^{\underline{n}}$ of $GL_n(\mathbb{A}_F)$ (conj.: $\longleftrightarrow$ N-dim$^{\underline{l}}$ irred. rep$^{\underline{n}}$ of L_F)

$\longrightarrow$ A. defines sign (π).

$\longrightarrow$ Define global parameters: formal sums $\boxplus_i \pi_i[d_i]$. $d_i \geq 1$.