

RANDOM MATRIX THEORY AND  
 $L$ -FUNCTIONS IN FUNCTION  
FIELDS

Júlio César Bueno de Andrade  
School of Mathematics

July 2012

A DISSERTATION SUBMITTED TO THE UNIVERSITY OF BRISTOL  
IN ACCORDANCE WITH THE REQUIREMENTS OF THE DEGREE  
OF DOCTOR OF PHILOSOPHY IN THE FACULTY OF SCIENCE



# Abstract

It is an important problem in analytic number theory to estimate mean values of the Riemann zeta-function and other  $L$ -functions. The study of moments of  $L$ -functions has some important applications, such as to give information about the maximal order of the Riemann zeta-function on the critical line, the Lindelöf Hypothesis for  $L$ -functions and non-vanishing results. Furthermore, according to the Katz-Sarnak philosophy [Katz-Sar99a, Katz-Sar99b] it is believed that the understanding of mean values of different families of  $L$ -functions may reveal the symmetry of such families.

The analogy between characteristic polynomials of random matrices and  $L$ -functions was first studied by Keating and Snaith [Kea-Sna00a, Kea-Sna00b]. For example, they were able to conjecture asymptotic formulae for the moments of  $L$ -functions in different families. The purpose of this thesis is to study moments of  $L$ -functions over function fields, since in this case the  $L$ -functions satisfy a Riemann Hypothesis and one may give a spectral interpretation for such  $L$ -functions as the characteristic polynomial of a unitary matrix. Thus, we expect that the analogy between characteristic polynomials and  $L$ -functions can be further understood in this scenario.

In this thesis, we study power moments of a family of  $L$ -functions associated with hyperelliptic curves of genus  $g$  over a fixed finite field  $\mathbb{F}_q$  in the limit as  $g \rightarrow \infty$ , which is the opposite limit considered by the programme of Katz and Sarnak. Specifically, we compute some average value theorems of  $L$ -functions of curves and we extend to the function field setting the heuristic for integral moments and ratios of  $L$ -functions previously developed by Conrey *et.al* [CFKRS05, Conr-Far-Zir] for the number field case.



*This thesis is dedicated to the memory  
of my father, Antônio de Andrade.*



# Acknowledgements

First of all I would like to thank God and Jesus, for conceding me the grace to write this thesis and finish my PhD studies.

This thesis would not have been possible without the help and guidance of my PhD supervisors, Jon Keating and Nina Snaithe. So I would like to begin by saying that I am very grateful to Jon for your continued support, trust, encouragement, patience and for all the time and care he invested in me and in my work. Jon taught me how to do research in mathematics and to think independently, his terrific intuition and knowledge always helped me when I found myself stuck on a problem. These last three years under Jon's supervision definitely made me a better mathematician. Thanks Jon.

In second, but not least, I would like to thank Nina for being so patient and very gentle with all the silly questions I have raised to her, for reading all my drafts and correct them, and for helping me with the preparation of all my talks presented in the course of my PhD.

I would also like to thank the professors Trevor Wooley, Pär Kurlberg, Tim Browning, Chantal David and Kannan Soundararajan for the many useful discussions that helped improve this thesis. I am especially grateful to Brian Conrey and Zeév Rudnick for many interesting discussions, for all the interesting problems that they presented me and for the help with my number theory questions.

A special mention goes to all my friends, first for all my Brazilian friends (especially Fábio, Marcos, Maurício, Raphael and Rodrigo) for staying in touch with me and letting me updated with the news from far, and in second for all

---

the friends that I have made here in UK and all fellow PhD students at our Department, a special thanks to Amy Mason, Alan Lee and Sandro Bettin for the many discussions.

I also owe a special thanks to all members of the Mathematics Department and from the University of Bristol in general. In particular Fionnuala Hill, Chrystal Cherniwchan and Briony Maitland for helping me with various issues concerning the university and the city of Bristol. I would like to thank the department as a whole for offering a warm and stimulating place, both for research and teaching, being my second home in these last years.

I wish to thank my former professors from São Paulo State University due to all learning received from them, especially to Alice, Jairo, Renata, Rômulo and Vanderlei. It is with great gratitude that I would like to mention the Professor Stavros Christodoulou who helped and advised my first steps in the study of analytic number theory and always has been interested and concerned about my academic education.

Many thanks also to all my family members, especially to my uncle Manoel, for helping me to come to the England. From all my family members my mother Inez deserves a special thanks because she always has been at my side, for teaching me that education and study should come first in our choices and to the support she always gave me for the achievement of all my dreams.

I would like to thank Brian Conrey and Zeév Rudnick, my examiners, for carefully reading the first version of this thesis and making valuable remarks and suggestions.

Although this thesis has my name written on it, if I could share this with another person and write one more name in this work, this name definitely would be Carolina, my wife and love of my life. I am very grateful to her for all the love, care, support, trust, perseverance, dedication and joyful moments. She has been the cornerstone of my life both in the happiest moments as in the most difficult ones over these last years. Thank you Carolina, this thesis would not have been possible without you.



---

My PhD was generously supported by an Overseas Research Scholarship and an University of Bristol Research Scholarship, both of which I am extremely grateful for allow me to obtain a PhD degree.



# Author's Declaration

I declare that the work in this thesis was carried out in accordance with the Regulations of the University of Bristol. The work is original except where indicated by special reference in the text. No part of the dissertation has been submitted for any other degree. Any views expressed in the dissertation are those of the author and do not necessarily represent those of the University of Bristol. The thesis has not been presented to any other university for examination either in the United Kingdom or overseas.

---

Júlio César Bueno de Andrade

Date: 31th May 2012



# Contents

<b>Abstract</b>	<b>ii</b>
<b>Acknowledgements</b>	<b>vi</b>
<b>Author's Declaration</b>	<b>x</b>
<b>Notation</b>	<b>xviii</b>
<b>1 Introduction</b>	<b>1</b>
1.1 Prime Numbers and the Riemann zeta–function . . . . .	1
1.1.1 Historical Introduction . . . . .	1
1.1.2 Zeros of the Riemann zeta–function, the Prime Number Theorem and the Riemann Hypothesis . . . . .	3
1.1.3 Moments of the Riemann zeta–function . . . . .	6
1.2 Analytic Theory of $L$ –functions . . . . .	10
1.2.1 Dirichlet $L$ –functions . . . . .	10
1.2.2 Average Value Theorems of Dirichlet $L$ –functions . . . . .	12
1.2.3 Quadratic Dirichlet $L$ –functions . . . . .	15
1.2.4 Mean Value Theorems of Quadratic Dirichlet $L$ –functions	16
1.3 Random Matrix Theory . . . . .	18
1.3.1 Random Matrix Theory and the Classical Compact Groups	18
1.3.2 Random Matrix Theory and Number Theory: The meet- ing between Dyson and Montgomery . . . . .	20
1.4 Families of $L$ –functions and the Katz–Sarnak Philosophy . . . . .	23

1.4.1	$n$ -level density . . . . .	24
1.4.2	$k$ th-consecutive spacings . . . . .	27
1.4.3	$k$ th-lowest eigenvalue . . . . .	27
1.5	Characteristic Polynomials and the Keating–Snaith Conjectures	28
1.5.1	Moments of $L$ -functions and Characteristic Polynomials	29
1.6	Overview of this Thesis . . . . .	32
<b>2</b>	<b>Function Field Preliminaries</b>	<b>33</b>
2.1	Polynomials over Finite Fields . . . . .	33
2.1.1	Basic facts about $\mathbb{F}_q$ and $\mathbb{F}_q[T]$ . . . . .	33
2.1.2	Prime Number Theorem in $A = \mathbb{F}_q[T]$ . . . . .	35
2.2	Dirichlet $L$ -functions over the rational function field $\mathbb{F}_q(T)$ . . .	36
2.2.1	Characters and the Reciprocity Law . . . . .	36
2.2.2	General Dirichlet $L$ -functions . . . . .	37
2.2.3	Quadratic Characters and the Corresponding $L$ -functions	39
2.2.4	Zeta functions associated with Curves . . . . .	40
2.2.5	Spectral Interpretation . . . . .	42
2.2.6	The Hyperelliptic Ensemble $\mathcal{H}_{2g+1,q}$ . . . . .	44
2.3	Average Value Theorems of $L$ -functions in Function Fields . . .	45
<b>3</b>	<b>The Mean Value of <math>L(\frac{1}{2}, \chi)</math> in the Hyperelliptic Ensemble</b>	<b>48</b>
3.1	Statement of the Theorem . . . . .	49
3.2	“Approximate” Functional Equation . . . . .	51
3.3	Setting Up The Problem . . . . .	52
3.3.1	Averaging the Approximate Functional Equation . . . . .	53
3.4	The Main Term Calculation . . . . .	53
3.4.1	Sieving out Square-Free and Coprime Polynomials. . . . .	54
3.4.2	A Sum Over Monic Polynomials. . . . .	57
3.4.3	Auxiliary Lemmas . . . . .	58
3.5	Estimating the Contributions of Non-Squares to the Average. . .	63
3.6	Proof of the Main Theorem . . . . .	65

<b>4</b>	<b>A Mean Value Theorem over Monic Irreducible Polynomials</b>	<b>67</b>
	in $\mathbb{F}_q[T]$	
4.1	Setting Up the Problem . . . . .	68
4.2	The Main Term Calculation . . . . .	70
4.3	Bounding Character Sums Over Primes . . . . .	72
4.4	Proof of the Main Theorem . . . . .	74
<b>5</b>	<b>Asymptotic Averages for the Class Numbers over Function</b>	<b>76</b>
	<b>Fields</b>	
5.1	The Class Number in the Number Field Setting . . . . .	76
5.2	The Function Field Case . . . . .	78
5.3	Statement of Results . . . . .	80
5.4	Preparation for the Proof of Main Theorems . . . . .	81
	5.4.1 Proof of Proposition 5.4.1 . . . . .	82
	5.4.2 Proof of Proposition 5.4.3 . . . . .	89
5.5	Proof of the Main Theorems . . . . .	94
<b>6</b>	<b>Integral Moments of <math>L</math>-functions in Function Fields</b>	<b>95</b>
6.1	Conjecture for Integral Moments of $L(\frac{1}{2}, \chi_D)$ over the rational function field . . . . .	96
6.2	Heuristic Derivation of the Conjecture . . . . .	100
	6.2.1 Some Analogies Between Classical $L$ -functions and $L$ - functions over Function Fields . . . . .	100
	6.2.2 Applying the Recipe for $L$ -Functions over Function Fields	101
6.3	Putting the Conjecture in a More Useful Form . . . . .	107
6.4	The Final Form of the Conjecture . . . . .	113
6.5	Some Conjectural Values for Moments of $L$ -functions in the Hyperelliptic Ensemble . . . . .	117
	6.5.1 First Moment . . . . .	117
	6.5.2 Second Moment . . . . .	121
	6.5.3 Third Moment . . . . .	123

6.6	Leading Order Asymptotic for the Moments of $L(s, \chi_D)$ . . . . .	124
<b>7</b>	<b>Autocorrelation of Ratios of <math>L</math>-functions over Rational Function Fields <math>\mathbb{F}_q(T)</math></b>	<b>127</b>
7.1	Conrey, Farmer and Zirnbauer's Recipe for Ratios of $L$ -Functions	127
7.1.1	The Recipe . . . . .	128
7.2	Autocorrelation of Ratios of Characteristic Polynomials . . . . .	130
7.3	Applying the Recipe for $L$ -functions over Function Fields. . . . .	131
7.4	Refinements of the Conjecture . . . . .	137
7.4.1	Closed form expression for $A_{\mathcal{D}}$ . . . . .	137
7.4.2	Combinatorial Sum as Multiple Integrals . . . . .	139
7.5	The Final Form of the Conjecture . . . . .	140
<b>8</b>	<b>An Application of the Ratios Conjecture of <math>L</math>-functions over Function Fields: One-Level Density</b>	<b>141</b>
8.1	Applying the Ratios Recipe . . . . .	141
8.2	Mean Value Theorem for the Logarithmic Derivative of $L(s, \chi_D)$	146
8.3	The One-Level Density Formula . . . . .	147
8.3.1	The Scaled One-Level Density . . . . .	149
<b>9</b>	<b>Conclusion and Further Questions</b>	<b>153</b>
<b>A</b>	<b>The Leading Order Term for the First Moment Using the Function Field Tauberian Theorem</b>	<b>156</b>
<b>B</b>	<b>Brief Review on Probability Theory</b>	<b>160</b>
<b>C</b>	<b>The Gamma function and the Barnes <math>G</math>-function</b>	<b>162</b>
C.1	Euler's Gamma function . . . . .	162
C.1.1	Properties of the Gamma function. . . . .	162
C.2	The Barnes $G$ -function . . . . .	163
C.2.1	Properties of the Barnes $G$ -function . . . . .	164



**Bibliography**

**166**



# Notation

## Matrix Groups

- $I_N$  Identity matrix in  $N$  dimensions.
- $U^\dagger$  Conjugate Transpose of matrix  $U$ .
- $U(N)$  Unitary group: set of  $N \times N$  complex matrices  $U$  satisfying  
 $UU^\dagger = U^\dagger U = I_N$ .
- $O(N)$  Orthogonal group: elements in  $U(N)$  with real entries.
- $SO(N)$  Special orthogonal group: elements in  $O(N)$  with determinant 1.
- $USp(2N)$  Unitary symplectic group: elements  $U \in U(2N)$  such that  
 $UJU^t = U$ , with  $J = \begin{bmatrix} 0 & I_N \\ -I_N & 0 \end{bmatrix}$
- $dU$  Haar measure on a compact group  $G = U(N)$ ,  $SO(2N)$  or  $USp(2N)$ .
- $\Lambda_U(s)$  Characteristic polynomial  $\Lambda_U(s) = \det(I - U^\dagger s)$  associated to the matrix  $U$ .

**Analysis and Number Theory**

- $O$   $f(x) = O(g(x))$ , if there is a constant  $c > 0$  such that  
 $|f(x)| \leq cg(x)$  for all  $x \geq x_0$ .
- $\ll$   $f(x) \ll g(x)$ , the Vinogradov symbol:  $f(x) = O(g(x))$ .
- $o$   $f(x) = o(g(x))$ :  $\lim f(x)/g(x) = 0$ .
- $\sim$   $f(x) \sim g(x)$ :  $\lim f(x)/g(x) = 1$ .
- $\asymp$   $f(x) \asymp g(x)$ : when both  $f(x) \ll g(x)$  and  $g(x) \ll f(x)$  hold.
- $\Omega$   $f(x) = \Omega(g(x))$ :  $\limsup f(x)/g(x) > 0$ .
- $\int_{(c)}$  to indicate that the line of integration is  $\Re(s) = c$ .

The arithmetic functions  $d(n)$ ,  $\Lambda(n)$ ,  $\mu(n)$  and  $\varphi(n)$ , the Euler totient function, are defined as usual (see, for example, [Tit, Chapter 1]). The function  $w(n)$  is the number of distinct prime factors of  $n$ . Generally, we let  $\varepsilon$  stand for an arbitrarily small number, not necessarily the same at each occurrence in the proof of a lemma or a theorem. We also let  $\delta$  be some small number.

**Function Fields**

- $\mathbb{F}_q$  The finite field with  $q$  elements.
- $\mathbb{F}_q[T]$  The polynomial ring over  $\mathbb{F}_q$ .
- $\mathbb{F}_q(T)$  The rational function field over  $\mathbb{F}_q$ .
- $\mathcal{H}_d$  The set of square-free monic polynomials of degree  $d$  in  $\mathbb{F}_q[T]$ .
- $|f|$  the measure of the size of the polynomial  $f$ .
- $P$  usually denotes a monic irreducible polynomial in  $\mathbb{F}_q[T]$ .





# Chapter 1

## Introduction

### 1.1 Prime Numbers and the Riemann zeta–function

#### 1.1.1 Historical Introduction

In 1859, Riemann [Riemann] revisited the Euler product formula as the starting point for his famous eight–page manuscript, introducing the Riemann zeta–function defined for  $\Re(s) > 1$  by the series

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad (1.1.1)$$

where  $s$  is a complex variable. By Cauchy’s integral test we can see that the sum defining the Riemann zeta–function (1.1.1) is convergent for  $\Re(s) > 1$ . The series is absolutely and uniformly convergent in the domain  $\Re(s) \geq 1 + \delta$ , for every  $\delta > 0$ . Hence  $\zeta(s)$  is holomorphic for  $\Re(s) > 1$  and we have the Euler product formula

$$\zeta(s) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad (1.1.2)$$

which was first discovered by Euler [Euler] in 1737.

The importance of  $\zeta(s)$  in the theory of prime numbers and in analytic number theory lies in the fact that the Euler product formula connects the

natural numbers with the prime numbers, and thus we expect that properties of prime numbers are encoded in the properties of the Riemann zeta–function. The study of the zeta–function began with Euler, where making use of  $\zeta(s)$  he showed that

$$\log \log x = \sum_{p \leq x} \frac{1}{p} + O(1), \quad (1.1.3)$$

which implies the infinity of prime numbers.

The next result that Riemann established in his paper is the analytic continuation of  $\zeta(s)$ . He showed that  $\zeta(s)$  can be meromorphically continued into the whole complex plane  $\mathbb{C}$  with a simple pole at  $s = 1$  with residue 1. The idea of the proof is to use the Jacobi theta–function (see [Tit, Section 2.6])

$$\theta(x) = \sum_{n \in \mathbb{Z}} e^{-n^2 \pi x} \quad (1.1.4)$$

and note that

$$\zeta(s) = \frac{\pi^{s/2}}{\Gamma\left(\frac{s}{2}\right)} \int_0^\infty x^{s/2-1} \psi(x) dx \quad (\Re(s) > 1), \quad (1.1.5)$$

where  $\Gamma(s)$  is the usual gamma function and

$$\psi(x) = \sum_{n \geq 1} e^{-n^2 \pi x}. \quad (1.1.6)$$

We have that

$$\psi(x) = \frac{\theta(x) - 1}{2}, \quad (1.1.7)$$

and making use of the Poisson summation formula we obtain that the function  $\theta(x)$  satisfies the following functional equation

$$\theta\left(\frac{1}{x}\right) = \sqrt{x} \theta(x). \quad (1.1.8)$$

Now, using (1.1.8) with (1.1.7) into (1.1.5) we obtain that

$$\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty (x^{-s/2-1/2} + x^{s/2-1}) \psi(x) dx. \quad (1.1.9)$$

The integral is convergent for all values of  $s$  since  $\psi(x)$  decreases more rapidly than any power of  $x$  for large  $x$ . Thus we can conclude that  $\zeta(s)$  is analytic in the whole complex plane, except for a simple pole at  $s = 1$ .



Riemann also deduced from (1.1.9) that

$$\xi(s) := \frac{s(s-1)}{2} \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \xi(1-s). \quad (1.1.10)$$

The formula above is called the *functional equation* of the Riemann zeta-function. To simplify the notation, the functional equation can be written as

$$\zeta(s) = \chi(s) \zeta(1-s), \quad (1.1.11)$$

where

$$\chi(s) = \pi^{s-1/2} \frac{\Gamma\left(\frac{1}{2} - \frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}. \quad (1.1.12)$$

**Remark 1.1.1.** *A fact that deserves to be taken in consideration is that in the sketch of the proof of the functional equation (1.1.10) for the  $\zeta(s)$  we made use of the theta-function  $\theta(x)$ , which is an example of a modular form, since it is periodic with period 1 and satisfies a transformation formula relating  $x \leftrightarrow \frac{1}{x}$ . Modular forms and Automorphic forms in general are connected with the **Langlands Program** which very briefly states that all reasonable generalizations of the Riemann zeta-function are somehow related to modular forms, Galois groups and general reciprocity laws in an appropriate meaning (see [Ber-Gel] and [Gel]).*

### 1.1.2 Zeros of the Riemann zeta-function, the Prime Number Theorem and the Riemann Hypothesis

We have the following integral representation for the Riemann zeta-function

$$\zeta(s) = \frac{e^{-i\pi s} \Gamma(1-s)}{2\pi i} \int_{\mathcal{C}} \frac{z^{s-1}}{e^z - 1} dz, \quad (1.1.13)$$

where  $\mathcal{C}$  is the contour that starts at  $-\infty$ , comes along the  $x$ -axis (a little below it), makes a small loop around the origin (counterclockwise) and returns to  $-\infty$  along the  $x$ -axis (just above it).

It follows from (1.1.13) that for all integers  $n \geq 0$

$$\zeta(-n) = -\frac{B_{n+1}(1)}{(n+1)}, \quad (1.1.14)$$

where the  $B_n$ 's are the Bernoulli numbers (see [Apostol, Section 12.12]). In particular,  $\zeta(-2k) = 0$  for all integers  $k \geq 1$ , these are called the *trivial zeros* of the Riemann zeta-function. Denoting  $s = \sigma + it$ , Riemann showed that  $\zeta(s)$  has no zeros in the half-plane  $\sigma > 1$  and that  $\zeta(s)$  vanishes infinitely often in the critical strip  $0 \leq \sigma \leq 1$ .

Riemann sketched a proof that if no zeros of  $\zeta(s)$  lie on the edge of the critical strip, i.e.,  $\Re(s) = 1$ , then the Prime Number Theorem (P.N.T.) should follow as a corollary of this fact. The Prime Number Theorem can be stated as

**Theorem 1.1.2** (Prime Number Theorem). *Let  $\pi(x) = \sum_{p \leq x} 1$ . Then*

$$\pi(x) \sim \frac{x}{\log x} \tag{1.1.15}$$

as  $x \rightarrow \infty$ .

The Prime Number Theorem was conjectured by Legendre [Legendre] and Gauss [Gauss] and for about 100 years eminent mathematicians tried to prove this conjecture, but only in 1896 J. Hadamard [Hadamard] and C. J. de la Vallée Poussin [Poussin] independently proved the validity of the conjecture.

Continuing his investigation, Riemann in his paper introduces the following function of complex variable  $t$

$$\xi(t) = \frac{1}{2} s(s-1) \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s), \tag{1.1.16}$$

where  $s = \frac{1}{2} + it$  and he was able to show that  $\xi(t)$  is an even and entire function whose zeros have the imaginary part between  $-i/2$  and  $i/2$ .

Now we are in a position to present the celebrated Riemann Hypothesis (R.H.)

**Conjecture 1.1.3** (Riemann Hypothesis). *All zeros of the function  $\xi(t)$  are real.*

In his paper [Riemann], Riemann conjectured that all zeros (the *nontrivial* zeros) of  $\zeta(s)$  in the critical strip lie on the symmetry line of the functional

equation, that is,  $\sigma = \frac{1}{2}$  (critical line). So we can state the Riemann Hypothesis as

**Conjecture 1.1.4** (Riemann Hypothesis). *The nontrivial zeros of  $\zeta(s)$  have real part equal to  $\frac{1}{2}$ .*

The Riemann Hypothesis still remains an open problem until the present day and in the opinion of many mathematicians is the most important open problem in mathematics. The Riemann Hypothesis is so important that the Clay Mathematics Institute has a prize of \$1,000,000 for a valid proof of it. For a description of the Riemann Hypothesis given by the Clay Mathematics Institute we suggest the section by E. Bombieri in [Car-Jaf-Wil].

Although no valid proof of the Riemann Hypothesis has been found, there is extensive numerical evidence that corroborate its validity (see [Lun-Rie-Win] and [Odl]). Hardy [Har-SZR] proved in 1914 that there are infinitely many zeros on the critical line and Selberg [Sel-ZR] proved that a positive proportion of the zeros lie on the critical line. This proportion has been improved to  $1/3$  by Levinson [Levinson], to  $2/5$  by Conrey [Conr-MTF] and recently due to Bui, Conrey and Young [Bui-Conr-Young] we know that more than 41% of the zeros are on the critical line.

We can say that one of the strongest pieces of evidence for the validity of the Riemann Hypothesis is the analogous theory of algebraic varieties over finite fields, due to the fact that for such varieties the associated Zeta function satisfies the analogue of the Riemann Hypothesis (see [Weil-CA], [Deligne I] and [Deligne II]).\*

To study the vertical distribution of zeros of  $\zeta(s)$ , let us define the following function

$$N(T) := \#\{\rho = \beta + i\gamma : 0 \leq \beta \leq 1, 0 \leq \gamma \leq T\}. \quad (1.1.17)$$

Making use of the argument principle it is known [VM] that

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O\left(\frac{1}{T}\right), \quad (1.1.18)$$

---

\*See next chapter for more details on this topic

where

$$S(T) = \frac{1}{\pi} \Im \left( \log \zeta \left( \frac{1}{2} + it \right) \right) \quad (1.1.19)$$

describes the fluctuations around the mean and is defined in such way that it varies continuously along the straight lines joining  $2$  to  $2 + iT$ , and  $2 + iT$  to  $1/2 + iT$  with initial value of  $0$ . When  $1/2 + iT$  is a Riemann zero,  $S(T)$  has a discontinuity jump. The following theorem was presented in Riemann's paper [Riemann], but the first proof was given by von Mangoldt [VM].

**Theorem 1.1.5** (The Riemann–von Mangoldt formula).

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T). \quad (1.1.20)$$

The connection between the zeros of  $\zeta(s)$  and the prime numbers appears in the error term of the Prime Number Theorem. Let us denote the nontrivial zeros by  $\rho_n = \beta_n + i\gamma_n$ , where  $\dots \gamma_{-1} < 0 < \gamma_1 \leq \gamma_2 \leq \dots$ . It is known that if

$$\Theta = \sup_{\rho} \Re(\rho) = \sup_n \beta_n \quad (1.1.21)$$

then

$$\pi(x) = \text{Li}(x) + O(x^{\Theta} \log x), \quad (1.1.22)$$

where  $\text{Li}(x)$  is the logarithmic integral and here  $1/2 \leq \Theta \leq 1$ . The Riemann hypothesis is equivalent to having  $\Theta = 1/2$ . Information about  $\Theta$  are not well known and the best zero-free region so far is due to Korobov and Vinogradov (for more details about zero-free regions see, [Tit, Chapter 6] and [Ivic, Chapter 6]).

### 1.1.3 Moments of the Riemann zeta-function

A very important problem in analytic number theory is to estimate moments of the Riemann zeta-function as  $T \rightarrow \infty$ . The  $2k^{\text{th}}$  moment of the Riemann zeta-function is defined by

$$M_k(T) := \frac{1}{T} \int_0^T |\zeta(\frac{1}{2} + it)|^{2k} dt. \quad (1.1.23)$$

The problem of moments of the Riemann zeta–function is connected with the famous Lindelöf hypothesis, which is the following conjecture

**Conjecture 1.1.6** (The Lindelöf Hypothesis). *For every  $\varepsilon > 0$*

$$\zeta\left(\frac{1}{2} + it\right) = O(t^\varepsilon). \quad (1.1.24)$$

The connection comes from the fact that the Lindelöf hypothesis is equivalent to the statement that  $M_k(T) = O(T^\varepsilon)$  for all positive integers  $k$  (see [Tit, Theorem 13.2]). Hence the problem about mean values of the modulus of Riemann zeta–function is related with many questions on  $O$ –results,  $\Omega$ –results and about the growth of the Riemann zeta–function on the critical line. We can say that problems involving moments of the Riemann zeta–function are considered one of the most profound and important topics in analytic number theory nowadays.

The leading order asymptotic for  $M_k(T)$  is known just for the second moment,  $k = 1$  and for the fourth moment,  $k = 2$ . The second moment is due to Hardy and Littlewood [Har-Lit],

$$M_1(T) \sim \log T. \quad (1.1.25)$$

Later, in 1926, Ingham [Ing] considered the asymptotic formula for the shifted second moment and derived

$$M_1(T) = \log \frac{T}{2\pi} + 2\gamma - 1 + O(T^{-1/2} \log T). \quad (1.1.26)$$

The error was improved by the work of Heath–Brown and Huxley [HB-Hux] to be  $O(T^{-15/22+\varepsilon})$  and by Watt [Watt] to be  $O(T^\theta \log^\varphi(T+2))$  with  $\varphi < 4$  and  $\theta = 131/416$ .

The calculations for the fourth moment are deeper and complicated, but in the same paper, Ingham established that

$$M_2(T) \sim \frac{1}{2\pi^2} (\log T)^4, \quad (1.1.27)$$

and the lower order terms were given by Heath–Brown [HB-FMRZ], who showed that

$$M_2(T) = \sum_{n=0}^4 c_n (\log T)^n + O(T^{-1/8+\varepsilon}), \quad (1.1.28)$$

where the  $c_n$ 's are absolute constants. The main term was also obtained by Conrey [Conr-NFPMR] as the residue of a certain function at  $s = 0$ .

Until the present day no other mean values of the Riemann zeta–function have been proved and it is possible that this problem may be beyond our reach with the available techniques. It was conjectured by Conrey and Ghosh [Conr-Gho98] that

$$M_3(T) \sim \frac{42}{9!} a(3) (\log T)^9, \quad (1.1.29)$$

and by Conrey and Gonek [Conr-Gon] that

$$M_4(T) \sim \frac{24024}{16!} a(4) (\log T)^{16}, \quad (1.1.30)$$

where

$$a_k = \prod_p \left[ \left(1 - \frac{1}{p}\right)^{k^2} \sum_{m=0}^{\infty} \left( \frac{\Gamma(m+k)}{m! \Gamma(k)} \right)^2 p^{-m} \right]. \quad (1.1.31)$$

For the other positive integer values of  $k$ , it is believed that, as  $T \rightarrow \infty$ , there is a positive constant  $c_k$  such that

$$M_k(T) \sim c_k (\log T)^{k^2}. \quad (1.1.32)$$

Due to the work of Conrey and Ghosh [Conr-Gho92] the conjecture above assumed a more explicit form, namely

$$c_k = \frac{a_k g_k}{\Gamma(k^2 + 1)}, \quad (1.1.33)$$

where

$$a_k = \prod_p \left[ \left(1 - \frac{1}{p}\right)^{k^2} \sum_{m \geq 0} \frac{d_k(p^m)^2}{p^m} \right], \quad (1.1.34)$$

$g_k$  is an integer when  $k$  is an integer and  $d_k(n)$  is the number of ways to represent  $n$  as a product of  $k$  factors. The main tool used by Conrey and Ghosh

in their paper is the Montgomery and Vaughan [Mon-Vau73, Mon-Vau74] mean value theorem, which can be stated as

$$\int_0^T \left| \sum_{n=1}^N a_n n^{it} \right|^2 dt = \sum_{n=1}^N (T + O(n)) |a_n|^2. \quad (1.1.35)$$

Although not much has been proved for  $k$  beyond 2, we still have interesting results on bounds for  $M_k(T)$ . Ramachandra [Ram-SRII] established the lower bound  $M_k(T) \gg (\log T)^{k^2}$  for positive integers  $2k$  and Heath–Brown [HB-FMR] established this for all positive rational numbers  $k$ . Under the R.H., Conrey and Ghosh [Conr-Gho84] showed that, for any fixed  $k \geq 0$ ,

$$M_k(T) \geq \left( \frac{a_k}{\Gamma(k^2 + 1)} + o(1) \right) (\log T)^{k^2}. \quad (1.1.36)$$

This result has been improved by Balasubramanian and Ramachandra [Bal-Ram] and Soundararajan [Sound-MVRZ]. Recently Soundararajan and Radziwill [Rad-Sound] obtained continuous lower bounds of the correct order of magnitude for the  $2k$ –th moment of the Riemann zeta function for all  $k \geq 1$ .

Considering now upper bounds for  $M_k(T)$  we have unconditionally  $M_k(T) \ll (\log T)^{k^2}$  for  $k = 1/n$ , where  $n$  is a positive integer [HB-FMR] and under the R.H., Ramachandra [Ram-SRII, Ram-SRIII] and Heath–Brown [HB-FMR, HB-FMR II] proved that the above is true for  $0 \leq k \leq 2$ . More recently Soundararajan [Sound-MRZ] under R.H. obtained that  $M_k(T) \ll (\log T)^{k^2 + \varepsilon}$  for every positive real number  $k$  and every  $\varepsilon > 0$ .

A different and important type of mean value theorem involving  $\zeta(s)$  which was first studied by Bohr and Landau [Boh-Lan] is the mollified moments of the Riemann zeta function

$$\frac{1}{T} \int_0^\infty |\zeta(\tfrac{1}{2} + it)|^{2k} |M_N(\tfrac{1}{2} + it)|^2 dt, \quad (1.1.37)$$

where

$$M_N(s) = \sum_{n=1}^N \frac{a(n)}{n^s}, \quad (1.1.38)$$

and the coefficients  $a(n)$  are carefully chosen to have a specific form similar to the Möbius function. Thus  $M_N(s)$  can be viewed as an approximation to

$1/\zeta(s)$  in the half plane  $\sigma > 1$ . If this approximation is also valid inside the critical strip, we expect that multiplying  $\zeta(s)$  by  $M_N(s)$  should absorb the large values of the zeta–function.

The calculations for the mollified second moment were established by Balasubramanian, Conrey and Heath–Brown [Bal-Conr-HB]. They showed that for  $N = T^\theta$  with  $\theta = 1/2 - \varepsilon$ , and  $a(n) \ll n^\varepsilon$ , then it follows that

$$\frac{1}{T} \int_0^T |\zeta M_N(\frac{1}{2} + it)|^2 dt \sim \sum_{h,k \leq N} \frac{a(h)\overline{a(k)}(h,k)}{hk} \left( \log \left( \frac{T(h,k)^2}{2\pi hk} \right) + 2\gamma - 1 \right). \quad (1.1.39)$$

It is from these mollification techniques that the theorems on the proportion of zeros in the critical line followed. When  $a(n)$  is similar to  $\mu(n)$ , using Kloosterman sums techniques, Conrey [Conr-MTF] increased the length of the polynomial to  $T^{4/7-\varepsilon}$ , this led to his theorem that more than 40% of the zeros of the Riemann zeta–function lie on the critical line as said earlier.

Through heuristic arguments Farmer [Far-LMR] suggested that Conrey’s result still holds when  $\theta$  is arbitrarily large. And as a consequence of Farmer’s conjecture we have that almost all the zeros lie on the critical line, which is a surprising consequence.

## 1.2 Analytic Theory of $L$ –functions

We will now present a generalization of the Riemann zeta–function. For further details and a deeper treatment of this subject see [Mon-Vau, Chapter 10], [Daven] and [Iwan-Kow, Chapter 5].

### 1.2.1 Dirichlet $L$ –functions

We will begin this section by defining the Dirichlet character.

**Definition 1.2.1.** *A Dirichlet character to modulus  $q$ , where  $q \in \mathbb{N}$ , is a function  $\chi : \mathbb{Z} \rightarrow \mathbb{C}$  such that*



(i)  $\chi(mn) = \chi(m)\chi(n)$  for all  $m, n \in \mathbb{Z}$ ,

(ii)  $\chi(n + mq) = \chi(n)$  for all  $m, n \in \mathbb{Z}$ ,

(iii)  $\chi(1) = 1$ , and

(iv)  $\chi(n) = 0$  whenever  $(n, q) \neq 1$ .

**Definition 1.2.2.** Let  $\chi$  be a Dirichlet character mod  $q$  and let  $d \mid q$ . The number  $d$  is called an induced modulus for  $\chi$  if

$$\chi(n) = 1 \quad \text{whenever} \quad (n, q) = 1 \quad \text{and} \quad n \equiv 1 \pmod{d}. \quad (1.2.1)$$

A Dirichlet character  $\chi$  modulo  $q$  is said to be *primitive* mod  $q$  if it has no induced modulus  $d < q$ .

**Definition 1.2.3.** Let  $\chi$  be a Dirichlet character modulo  $q$ . The Dirichlet  $L$ -function corresponding to  $\chi$  is defined to be

$$L(s, \chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}. \quad (1.2.2)$$

The Dirichlet  $L$ -function is absolutely convergent for  $\Re(s) > 1$  and locally uniformly convergent, so  $L(s, \chi)$  is holomorphic in the same region and if we call  $\chi_0$  to be the principal character modulo  $q$ , where  $\chi_0(n) = 1$  if  $(n, q) = 1$  and is 0 otherwise, we have that for  $\chi \neq \chi_0$  the series defines a holomorphic function for  $\Re(s) > 0$ . Since characters are multiplicative we can deduce in the same way that is done for the Riemann zeta function that

$$L(s, \chi) = \prod_{p \text{ prime}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}. \quad (1.2.3)$$

Restricting  $\chi$  to be primitive and defining the completed  $L$ -function by

$$\Lambda\left(\frac{1}{2} + s, \chi\right) := \left(\frac{q}{\pi}\right)^{s/2} \Gamma\left(\frac{s + \frac{1}{2} + \mathbf{a}}{2}\right) L\left(\frac{1}{2} + s, \chi\right), \quad (1.2.4)$$

where

$$\mathbf{a} = \begin{cases} 0 & \text{if } \chi(-1) = 1, \\ 1 & \text{if } \chi(-1) = -1, \end{cases}$$

we obtain that (1.2.4) is an entire function and satisfies the following functional equation

$$\Lambda\left(\frac{1}{2} + s, \chi\right) = \frac{\tau(\chi)}{i^a \sqrt{q}} \Lambda\left(\frac{1}{2} - s, \bar{\chi}\right), \quad (1.2.5)$$

where  $\tau(\chi)$  is the Gauss sum defined to be

$$\tau(\chi) := \sum_{n=1}^q \chi(n) e(n/q), \quad (1.2.6)$$

where  $e(x) := \exp(2\pi i x)$ . For a proof of these facts see [Daven, Chapter 9].

The functional equation shows that  $L(s, \chi)$  has an analytic continuation to the complex plane  $\mathbb{C}$  and is regular everywhere. The zeros of  $\Lambda(s, \chi)$  are located in the critical strip and this fact follows from the functional equation and the Euler product for  $L(s, \chi)$ , these zeros are called the nontrivial zeros of the  $L$ -function. We have the following conjecture

**Conjecture 1.2.4** (Grand Riemann Hypothesis). *All non-trivial zeros of Dirichlet  $L$ -functions lie on the critical line.*

Analogous to the Riemann zeta-function we can define the zero counting function by

$$N(T, \chi) := \#\{\beta + i\gamma : \Lambda(\beta + i\gamma) = 0, |\gamma| \leq T\}, \quad (1.2.7)$$

and for  $T \geq 2$  we have the analogue of the Riemann-von Mangoldt formula (see [Mon-Vau, Chapter 14]),

$$\frac{1}{2} N(T, \chi) = \frac{T}{2\pi} \log \frac{qT}{2\pi} - \frac{T}{2\pi} + O(\log qT). \quad (1.2.8)$$

## 1.2.2 Average Value Theorems of Dirichlet $L$ -functions

Similarly to the case of the Riemann zeta-function, the problem related to the calculations of mean values for Dirichlet  $L$ -functions is also an important and central problem in analytic number theory. We define the  $2k^{\text{th}}$  power moment of  $L(s, \chi)$  at the centre of the critical strip, that is, at  $s = 1/2$  to be

$$\frac{1}{\varphi^*(q)} \sum_{\chi \pmod{q}}^* |L\left(\frac{1}{2}, \chi\right)|^{2k}, \quad (1.2.9)$$

where  $\varphi^*(q)$  is the number of primitive characters and  $\sum^*$  denotes a sum over all primitive characters  $\chi$  modulo  $q$ . We will be concerned with the asymptotic behaviour of such moments in the limit as  $q \rightarrow \infty$ , this is the  $q$ -analogue of the moments of the Riemann zeta-function on the critical line.

The leading order asymptotic for (1.2.9) is known for the second moment,  $k = 1$ , due to Paley [Paley],

$$\frac{1}{\varphi^*(q)} \sum_{\chi(\bmod q)}^* |L(\tfrac{1}{2}, \chi)|^2 \sim \frac{\varphi(q)}{q} \log q. \quad (1.2.10)$$

Later, Iwaniec and Sarnak [Iwan-Sar] found an asymptotic formula with a power saving for the error term. And in 1981, Heath-Brown [HB-FMDL] established an asymptotic formula for the fourth power moment provided that  $q$  has not too many prime factors and therefore the error term below is dominated by the main term,

$$\frac{1}{\varphi^*(q)} \sum_{\chi(\bmod q)}^* |L(\tfrac{1}{2}, \chi)|^4 = \frac{1}{2\pi^2} \prod_{p|q} \frac{(1-p^{-1})^3}{(1+p^{-1})} (\log q)^4 + O\left(\frac{2^{\omega(q)} q (\log q)^3}{\varphi^*(q)}\right). \quad (1.2.11)$$

Still on the fourth moment, Soundararajan [Sound-FMDL] established the following asymptotic formula as  $q \rightarrow \infty$ ,

$$\frac{1}{\varphi^*(q)} \sum_{\chi(\bmod q)}^* |L(\tfrac{1}{2}, \chi)|^4 \sim \frac{1}{2\pi^2} \prod_{p|q} \frac{(1-p^{-1})^3}{(1+p^{-1})} (\log q)^4. \quad (1.2.12)$$

Recently, Young [Young-FMDL] obtained an asymptotic formula with a power saving when  $q \neq 2$  and  $q$  is prime,

$$\frac{1}{\varphi^*(q)} \sum_{\chi(\bmod q)}^* |L(\tfrac{1}{2}, \chi)|^4 = \sum_{i=0}^4 c_i (\log q)^i + O(q^{-5/512+\varepsilon}), \quad (1.2.13)$$

where  $c_i$  are computable constants and the exponent  $-5/512$  is given by the best-known bound on the size of the Hecke eigenvalue  $\lambda(n)$  associated with a Maass form due to Kim and Sarnak [Kim-Sar].

Conrey, Iwaniec and Soundararajan [Conr-Iwan-Sound] recently obtained the following asymptotic formula for a different average corresponding to the

sixth power moment

$$\begin{aligned} & \sum_{q \leq Q} \sum_{\chi \bmod q}^* \int_{-\infty}^{\infty} |\Lambda(\tfrac{1}{2} + iy, \chi)|^6 dy \\ & \sim 42a_3 \sum_{q \leq Q} \prod_{p|q} \frac{(1 - \frac{1}{p})^5}{(1 + \frac{4}{p} + \frac{1}{p^2})} \varphi^*(q) \frac{(\log q)^9}{9!} \int_{-\infty}^{\infty} \left| \Gamma\left(\frac{1/2 + iy}{2}\right) \right|^6 dy, \end{aligned} \quad (1.2.14)$$

where  $\chi$  is a primitive even Dirichlet character modulo  $q$ , i.e.  $\chi(-1) = 1$ ,  $a_3$  is a certain product over primes and  $\varphi^*(q)$  is the number of even primitive Dirichlet characters and the sum here is restricted to even primitive Dirichlet characters.

A more general problem involving moments of Dirichlet  $L$ -functions is to consider the  $q$ -aspect and the  $t$ -aspect and investigate asymptotic formulas or uniform estimates for both  $q$  and  $T$  in

$$\frac{1}{\varphi^*(q)} \sum_{\chi \bmod q}^* \int_0^T |L(\tfrac{1}{2} + it, \chi)|^{2k} dt, \quad (1.2.15)$$

see Montgomery [Mon-TMNT] and Motohashi [Mot-NMVL II, Mot-NMVL III] for some results involving this kind of average value. Recently, Bui and Heath-Brown [Bui-HB] showed that for  $q, T \geq 2$

$$\begin{aligned} & \sum_{\chi \bmod q}^* \int_0^T |L(\tfrac{1}{2} + it, \chi)|^4 dt \\ & = \left(1 + O\left(\frac{\omega(q)}{\log q} \sqrt{\frac{q}{\phi(q)}}\right)\right) \frac{\varphi^*(q)T}{2\pi^2} \prod_{p|q} \frac{(1 - p^{-1})^3}{(1 + p^{-1})} (\log qT)^4 \\ & \quad + O(qT(\log qT)^{7/2}), \end{aligned} \quad (1.2.16)$$

where the sum is over all primitive Dirichlet characters  $\chi$  modulo  $q$ ,  $\omega(q)$  is the number of distinct prime factors of  $q$  and  $\varphi^*(q)$  is the number of primitive Dirichlet characters.

Analogous to the case of the Riemann zeta-function, no other asymptotic formula is known in this case for  $k > 2$ . But we can investigate bounds for these moments and heuristic arguments suggest that the correct order of magnitude of (1.2.9) is  $\asymp (\log q)^{k^2}$ .

We have the following lower bound given by Rudnick and Soundararajan [Rud-Sound05] for all large primes  $q$ ,

$$\frac{1}{\varphi^*(q)} \sum_{\chi(\bmod q)}^* |L(\tfrac{1}{2}, \chi)|^{2k} \gg_k (\log q)^{k^2}. \quad (1.2.17)$$

And on the upper bounds we have the following results obtained by Huxley [Hux-MHZ]

$$\sum_{q \leq Q} \sum_{\chi(\bmod q)}^* |L(\tfrac{1}{2}, \chi)|^6 \ll Q^2 (\log Q)^9, \quad (1.2.18)$$

and

$$\sum_{q \leq Q} \sum_{\chi(\bmod q)}^* |L(\tfrac{1}{2}, \chi)|^8 \ll Q^2 (\log Q)^{16}, \quad (1.2.19)$$

which agree with the order of magnitude suggested above.

### 1.2.3 Quadratic Dirichlet $L$ -functions

We start this section by defining fundamental discriminants and the  $L$ -functions associated with real characters.

**Definition 1.2.5.** *The number  $d \neq 1$  is called a fundamental discriminant if either  $d \equiv 1 \pmod{4}$ ,  $d$  square-free, or  $d = 4N$ , where  $N$  is square-free and  $N \equiv 2, 3 \pmod{4}$ .*

We now denote  $\chi_d$  to be the Dirichlet character defined by the Kronecker's symbol  $\chi_d(n) = \left(\frac{d}{n}\right)$  with  $d$  being restricted to fundamental discriminants. The character  $\chi_d$  defined in this way only takes values  $-1, 0$  or  $1$  and if  $d > 0$  then  $\chi_d$  is called an even character, i.e.,  $\chi_d(-1) = 1$ , and if  $d < 0$  it is called an odd character, i.e.,  $\chi_d(-1) = -1$ . We present in the next theorem some properties of  $\chi_d$ , for details see [Mon-Vau].

**Theorem 1.2.6.**

*(i) Let  $d$  be a fundamental discriminant. Then  $\chi_d(n)$  is a primitive quadratic character modulo  $|d|$ . Also there is only one real primitive character modulo  $|d|$  for any such  $d$ .*

(ii) Any character  $\chi_d$  can be derived as a product of the characters  $\chi_{-4}$ ,  $\chi_8$ ,  $\chi_p$  and  $\chi_{-p}$ , where  $p \equiv 1 \pmod{4}$  for  $\chi_p$  and  $p \equiv 3 \pmod{4}$  for  $\chi_{-p}$  where  $p$  is odd. Remember that  $\chi_{\pm p}(n) = \left(\frac{n}{p}\right)$  is the Legendre symbol.

Now, consider the  $L$ -function associated with the character  $\chi_d$ , this  $L$ -function is usually called a quadratic Dirichlet  $L$ -function and has the following series representation and Euler product

$$L(s, \chi_d) = \sum_{n=1}^{\infty} \frac{\chi_d(n)}{n^s} \quad \text{and} \quad L(s, \chi_d) = \prod_p \left(1 - \frac{\chi_d(p)}{p^s}\right)^{-1}, \quad (1.2.20)$$

where  $\chi_d(n)$  is defined as above. The Dirichlet series is absolutely convergent for  $\Re(s) > 1$ , and  $\chi_d(n)$  is a primitive character for the fundamental discriminants  $d$ . We have that the quadratic Dirichlet  $L$ -functions have a functional equation and can be analytically continued to the whole complex plane.

Just as we proceed in the case of the Riemann zeta-function and Dirichlet  $L$ -functions we can do for quadratic Dirichlet  $L$ -functions. An important observation is the possibility of quadratic Dirichlet  $L$ -functions having zeros on the real axis. We recommend the reader to see [Conr-Sound], [Iwan-CEC] or [Mon-Vau, Chapter 11] for a detailed discussion on this subject.

### 1.2.4 Mean Value Theorems of Quadratic Dirichlet $L$ -functions

The problem of mean values for quadratic Dirichlet  $L$ -functions is to understand the asymptotic behavior of

$$\sum_{0 < d \leq D} L\left(\frac{1}{2}, \chi_d\right)^k, \quad (1.2.21)$$

as  $D \rightarrow \infty$ . In this context Jutila [Jutila] proved that

$$\begin{aligned} & \sum_{0 < d \leq D} L\left(\frac{1}{2}, \chi_d\right) \\ &= \frac{P(1)}{4\zeta(2)} D \left\{ \log(D/\pi) + \frac{\Gamma'}{\Gamma}(1/4) + 4\gamma - 1 + 4\frac{P'}{P}(1) \right\} + O(D^{3/4+\epsilon}) \end{aligned} \quad (1.2.22)$$

where

$$P(s) = \prod_p \left( 1 - \frac{1}{(p+1)p^s} \right).$$

The asymptotic formula for this first moment was previously conjectured by Goldfeld and Viola [Gold-Vio]. In the same paper [Jutila], Jutila also established the second moment of  $L(s, \chi_d)$  at the critical point  $s = \frac{1}{2}$ ,

$$\sum_{0 < d \leq D} L\left(\frac{1}{2}, \chi_d\right)^2 = \frac{c}{\zeta(2)} D \log^3 D + O(D(\log D)^{5/2+\varepsilon}) \quad (1.2.23)$$

with

$$c = \frac{1}{48} \prod_p \left( 1 - \frac{4p^2 - 3p + 1}{p^4 + p^3} \right).$$

Restricting  $d$  to be odd, square-free and positive, so that  $\chi_{8d}$  are real, primitive characters with conductor  $8d$  and with  $\chi_{8d}(-1) = 1$ , Soundararajan [Sound-NQDL] showed that

$$\begin{aligned} & \frac{1}{D^*} \sum_{0 < d \leq D}^* L\left(\frac{1}{2}, \chi_{8d}\right)^3 \\ & \sim \frac{1}{184320} \prod_{p \geq 3} \left( 1 - \frac{12p^5 - 23p^4 + 23p^3 - 15p^2 + 6p - 1}{p^6(p+1)} \right) (\log D)^6, \end{aligned} \quad (1.2.24)$$

where the sum  $\sum^*$  over  $d$  indicates that  $d$  is odd and square-free and  $D^*$  is the number of such  $d$  in  $(0, D]$ . In the same paper [Sound-NQDL], Soundararajan also gave a conjecture for the fourth moment. A few years later Diaconu, Goldfeld and Hoffstein [Diac-Gold-Hoff] showed a different way to obtain the same asymptotic formula using multiple Dirichlet series techniques. Recently, Young [Young-FMQDL] considered the smoothly weighted first moment of primitive quadratic Dirichlet  $L$ -functions and was able to obtain an error term which is the square root of the main term,

$$\sum_{(d,2)=1}^* L\left(\frac{1}{2}, \chi_{8d}\right) \Phi\left(\frac{d}{D}\right) = DP(\log D) + O(D^{1/2+\varepsilon}), \quad (1.2.25)$$

where  $\Phi$  is a smooth function of compact support and  $P$  is a linear polynomial depending on  $\Phi$ . Before finish writing this thesis, Young [Young-TMQDL]

posted a paper in the arXiv where he obtains the smooth third moment of quadratic Dirichlet  $L$ -functions with an error term of size  $O(D^{3/4+\varepsilon})$ ,

$$\sum_{(d,2)=1}^* L\left(\frac{1}{2}, \chi_{8d}\right)^3 F(d) = \sum_{(d,2)=1}^* P(\log d)F(d) + O(D^{3/4+\varepsilon}), \quad (1.2.26)$$

where  $F$  is a smooth, compactly-supported function on the positive reals with support in a dyadic interval  $[D/2, 3D]$  and satisfying  $F^j(x) \ll_j D^{-j}$  for  $j = 1, 2, \dots$ . The sum is over square-free numbers  $d$  and  $P(x)$  is a particular degree 6 polynomial.

Heuristic arguments suggest that the correct order of magnitude of the  $k^{\text{th}}$  moment is  $\asymp (\log D)^{k(k+1)/2}$  and the conjectured lower bound was proved by Rudnick and Soundararajan [Rud-Sound06] for every even natural  $k$

$$\sum_{|d|\leq D}^* L\left(\frac{1}{2}, \chi_d\right)^k \gg_k D(\log D)^{k(k+1)/2}. \quad (1.2.27)$$

## 1.3 Random Matrix Theory

Roughly speaking, Random Matrix Theory (RMT) is the study of matrices whose elements are random variables and is primarily concerned with the probabilistic properties of its eigenvalues and eigenvectors. For a complete treatment of the random matrix theory we suggest Mehta's book [Mehta]. For a review of the subject see [Tra-Wid], for a historical development of the theme we suggest [For-Sna-Ver] and for applications of the theory of random matrices in several areas we suggest the recent book [Ake-Baik-DiFran].

### 1.3.1 Random Matrix Theory and the Classical Compact Groups

By using random matrices to study spectra of heavy atoms in physics, Wigner [Wigner] investigated the *Gaussian Ensembles* which are (Gaussian Unitary-GUE, Orthogonal-GOE and Symplectic Ensembles-GSE), but for comparisons



and analogies with number theory, specifically with  $L$ -functions, the matrices which has proven to be useful are those associated with the three Classical Compact Groups. For completeness we will now present the Classical Compact Groups.

1. **Unitary Group**  $U(N)$ . Is the group of all  $N \times N$  matrices  $A$  such that  $AA^\dagger = A^\dagger A = I_N$ , where  $A^\dagger$  is the complex transpose of  $A$  and  $I_N$  is the  $N \times N$  identity matrix.

2. **Unitary Symplectic Group**  $USp(2N)$ . Is the group of all  $2N \times 2N$  matrices  $A \in U(N)$ , where  $A$  satisfy the condition  $AJA^t = A$  with

$$J = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}. \quad (1.3.1)$$

3. **Special Orthogonal Group**  $SO(N)$ . Is the group of  $N \times N$  matrices  $A \in U(N)$ , such that  $A^t A = AA^t = I_N$  with  $\det(A) = 1$ .

Each of the above groups is a compact Lie group and so they have a unique invariant probability measure under the action of the group, called the *Haar measure*. The Haar measure is different for each of these groups, for example, the Haar measure  $dA$  for  $U(N)$  and  $USp(2N)$  are

1.  $U(N)$

$$dA = \frac{1}{N!(2\pi)^N} \prod_{1 \leq j < k \leq N} |e^{i\theta_k} - e^{i\theta_j}|^2 d\theta_1 \dots d\theta_N \quad (1.3.2)$$

2.  $USp(2N)$

$$dA = \frac{1}{2^N (2\pi)^N N!} \prod_{1 \leq j < k \leq N} |e^{i\theta_k} - e^{i\theta_j}|^2 |e^{i\theta_k} - e^{-i\theta_j}|^2 \times \prod_{k=1}^N |e^{i\theta_k} - e^{-i\theta_k}| d\theta_1 \dots d\theta_N, \quad (1.3.3)$$

where  $e^{i\theta_n}$  are the eigenvalues of the associated matrices. Since the matrices of the classical groups are unitary matrices we have that their eigenvalues lie on

the unit circle. And for matrices in  $USp(2N)$  the eigenvalues occur in conjugate pairs and can be written as  $e^{i\theta_n}, e^{-i\theta_n}$  for  $1 \leq n \leq N$ . See Weyl [Weyl-CG] for a complete discussion about Haar measure and joint probability density of eigenvalues in the classical compact groups.

### 1.3.2 Random Matrix Theory and Number Theory: The meeting between Dyson and Montgomery

In the early 1970's Hugh Montgomery was visiting the Institute for Advanced Study in Princeton when he was introduced to Freeman Dyson. Montgomery was working on the *pair correlation* of the zeros of the Riemann zeta-function, i.e., how the distance between two zeros of the Riemann zeta-function behaves on average. At this meeting Montgomery showed such calculations to Dyson, who instantly realized that this pair correlation were similar to that of the eigenvalues of a random unitary matrix. The opportunity of this meeting, the Montgomery's calculations and the insight of Dyson on this question was the key point for the birth of the interaction between random matrices and the distribution of zeros of the Riemann zeta-function as quoted in [Dyson72].

Assuming the truth of the R.H., Montgomery [Mon-PCZR] stated the following conjecture

**Conjecture 1.3.1** (Pair Correlation of Zeros). *For fixed  $0 < a < b < \infty$  and  $\gamma, \gamma'$  generic ordinates of zeros of the Riemann zeta-function, we have*

$$\sum_{\substack{\gamma, \gamma' \in [0, T] \\ a \leq (\gamma - \gamma') \frac{\log T}{2\pi} \leq b}} 1 \sim \frac{T}{2\pi} \log T \left( \int_a^b 1 - \left( \frac{\sin(\pi u)}{\pi u} \right)^2 du + \delta(a, b) \right) \quad (1.3.4)$$

as  $T \rightarrow \infty$ . Where  $\delta(a, b) = 1$  if  $0 \in [a, b]$  and  $\delta(a, b) = 0$  otherwise.

Let  $f(x)$  be a suitable test function in the Schwarz class with the support of its Fourier transform  $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e(-x\xi)dx$  contained in  $(-1, 1)$ . Then we can now state the Montgomery–Odlyzko conjecture

**Conjecture 1.3.2** (Montgomery–Odlyzko). *If  $f(x)$  is a nice test function as above and under the same hypothesis as in Conjecture 1.3.1, such that  $f(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ , we have*

$$\begin{aligned} \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{n,m \leq T} f\left(\frac{\gamma_n \log \gamma_n}{2\pi} - \frac{\gamma_m \log \gamma_m}{2\pi}\right) \\ = \int_{-\infty}^{\infty} f(x) \left( \delta(x) + 1 - \left(\frac{\sin \pi x}{\pi x}\right)^2 \right) dx. \end{aligned} \quad (1.3.5)$$

Now, looking at the RMT side, for an  $N \times N$  unitary matrix with normalised eigenangles  $\theta_1, \dots, \theta_N$  the *two-point correlation function* for the matrix  $A \in U(N)$  is defined by Keating [Mez-Sna, pg 253] to be

$$R_2(A; x) := \frac{1}{N} \sum_{n=1}^N \sum_{m=1}^N \sum_{k=-\infty}^{\infty} \delta(x + Nk - \theta_n + \theta_m). \quad (1.3.6)$$

So we have that  $R_2(A; x)$  is  $N$ -periodic and for a suitable test function  $f$

$$\frac{1}{N} \sum_{n,m} f(\theta_n - \theta_m) = \int_{-N/2}^{N/2} R_2(A; x) f(x) dx. \quad (1.3.7)$$

In 1962 Dyson [Dyson] showed that

**Theorem 1.3.3.** *For a nice test function  $f$ , such that  $f(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ , we have*

$$\begin{aligned} \int_{G(N)} \frac{1}{N} \sum_{n,m \leq N} f(\theta_n - \theta_m) dA &= \int_{G(N)} \int_{-N/2}^{N/2} f(x) R_2(A; x) dx dA \\ &= \int_{-N/2}^{N/2} f(x) \left( \sum_{k=-\infty}^{\infty} \delta(x - kN) + 1 - \frac{\sin^2(\pi x)}{N^2 \sin^2(\frac{\pi x}{N})} \right) dx, \end{aligned} \quad (1.3.8)$$

and hence

$$\begin{aligned} \lim_{N \rightarrow \infty} \int_{G(N)} \frac{1}{N} \sum_{n,m \leq N} f(\theta_n - \theta_m) dA &= \lim_{N \rightarrow \infty} \int_{G(N)} \int_{-N/2}^{N/2} f(x) R_2(A; x) dx dA \\ &= \int_{-\infty}^{\infty} f(x) \left( \delta(x) + 1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2 \right) dx, \end{aligned} \quad (1.3.9)$$

where  $G(N) = U(N)$ ,  $USp(2N)$  or  $SO(2N)$ .

Thus comparing (1.3.5) with (1.3.9) we can conclude as did Dyson, that the pair correlation for  $U(N)$  is the same as the conjecture for the pair correlation for zeros of the Riemann zeta-function.

There is a considerable amount of numerical evidence provided by Odlyzko [Odl89] and Rubinstein [Rub-ESIZ] supporting the above conjecture. Besides the two-point correlation investigated above, higher correlations were also studied. For the three point correlation of the zeros of the Riemann zeta-function, Hejhal [Hejhal] showed that it is asymptotically the same as the three point correlation for  $U(N)$ . Generalizing the two-point and three-point correlation to all  $n$ -point correlations, Rudnick and Sarnak [Rud-Sar] showed that the  $n$ -point correlation of zeros of principal  $L$ -functions match the RMT analogous theorem for  $n$ -point correlation function. It is important to note that both works of Hejhal and Rudnick & Sarnak are based on appropriate restrictions on the support of the Fourier transform of the test function.

Using heuristic arguments Bogomolny and Keating [Kea93, Bog-Kea95, Bog-Kea96a, Bog-Kea96b] obtained the above conjectures on the  $n$ -point correlation of zeros of the Riemann zeta-function. An important fact is that the results derived by Bogomolny and Keating do not present any restriction on the support of the Fourier transform. The heuristic arguments are based on the following conjecture due to Hardy-Littlewood [Har-Lit23]

**Conjecture 1.3.4.** *Let  $\pi_2(k; X)$  be the number of primes  $p \leq X$  such that  $p + k$  is also a prime. Then,*

$$\pi_2(x) \sim \frac{X}{\log^2 X} C(k), \quad (1.3.10)$$

as  $X \rightarrow \infty$  and  $C(k) = 0$  if  $k$  is odd and for  $k$  even is

$$C(k) = 2 \prod_{\substack{q > 2 \\ q \text{ prime}}} \left(1 - \frac{1}{(q-1)^2}\right) \prod_{\substack{p > 2 \\ p|k \\ p \text{ prime}}} \left(\frac{p-1}{p-2}\right). \quad (1.3.11)$$

The formula given by Bogomolny and Keating [Bog-Kea96a] for the pair correlation includes all of the lower order terms that arise from arithmetical

considerations. Bogomolny and Leboeuf [Bog-Leb] have extended the same heuristic arguments and methods beyond the Riemann zeta-function case to compute the pair correlation for Dirichlet  $L$ -functions. Their result also matches with the  $U(N)$  RMT pair-correlation.

## 1.4 Families of $L$ -functions and the Katz–Sarnak Philosophy

One of the most promising ways to find a proof for the Riemann Hypothesis is due to what is called *Hilbert–Pólya conjecture* [Edwards, Ber-Kea]. The conjecture states that there is a self-adjoint (hermitian) operator whose eigenvalues are the nontrivial zeros of the Riemann zeta-function. There is a great deal of evidence for the validity of this conjecture, such as: numerical and theoretical calculations of the local spacing distribution between the high zeroes of  $L$ -functions, study of low-lying zeros of zeta functions and the eigenvalue distribution laws. But the strongest piece of evidence are the function field analogues<sup>†</sup> for zeta functions of curves over finite fields and varieties in general, since in this case we are able to provide a spectral interpretation of the zeros in terms of eigenvalues of Frobenius on cohomology.

Katz and Sarnak in [Katz-Sar99b] have presented the idea that for each family of  $L$ -functions there is a corresponding symmetry group, where the particular group is determined by symmetries of the family. This idea is based on the study of the distribution of zeros of zeta functions in the function field setting, as presented in their monograph [Katz-Sar99a].

The idea of symmetry types and families of  $L$ -functions were also exploited by Conrey and Farmer<sup>‡</sup> [Conr-Far], who presented the following conjecture concerning the moments of  $L$ -functions

---

<sup>†</sup>We will discuss this topic in more detail in Chapter 2.

<sup>‡</sup>We will explore this conjecture in the section 1.5.1.

**Conjecture 1.4.1** (Conrey–Farmer). *For some  $a_k$ ,  $g_k$  and  $B(k)$  we have,*

$$\frac{1}{\mathcal{Q}^*} \sum_{\substack{f \in \mathcal{F} \\ c(f) \leq \mathcal{Q}}} V(L_f(\frac{1}{2}))^k \sim \frac{a_k g_k}{\Gamma(1 + B(k))} (\log \mathcal{Q})^{B(k)}, \quad (1.4.1)$$

where the  $L$ -function have a functional equation relating  $s \leftrightarrow 1 - s$ , the family  $\mathcal{F}$  is partially ordered by the conductor  $c(f)$  and  $\mathcal{Q}^* = \#\{f : c(f) \leq \mathcal{Q}\}$ . Also we have that  $V(z)$  depends on the symmetry type of the family ( $V(z) = |z|^2$  for unitary symmetry and  $V(z) = z$  for orthogonal or symplectic symmetry),  $g_k$  and  $B(k)$  depend only on the symmetry type of the family of  $L$ -functions and are integral for integral  $k$  and the parameter  $a_k$  depends on the family and is computable in any specific case.

It is from the Katz-Sarnak philosophy [Katz-Sar99a, Katz-Sar99b] that the connection between random matrices and statistical properties of families of  $L$ -functions becomes more apparent, they showed that the distribution of the zeros are related to average over the classical compact groups and was conjectured by Katz–Sarnak that the zero statistic around the critical point of families of  $L$ -functions are related to the eigenvalue statistics associated with matrices of the classical compact groups near to the symmetry point for this case. For completeness, we now present other statistics involving eigenvalues of random matrices and zeros of  $L$ -functions and some results due to Katz and Sarnak.

### 1.4.1 $n$ -level density

We now introduce other statistic of the eigenvalues associated with matrices from the classical compact groups, called the  $n$ -level density. We refer the reader to the section of Conrey in [Mez-Sna]. Let us establish some notation that will be used in this section.

The sine ratios are

$$S(x) = \frac{\sin \pi x}{\pi x}, \quad (1.4.2)$$

and

$$S_N(x) = \frac{\sin Nx/2}{\sin x/2}. \quad (1.4.3)$$

The kernel functions are

$$K_{U(N)}(x, y) = S_N(y - x) \quad (1.4.4)$$

$$K_{SO(2N)}(x, y) = \frac{S_{2N-1}(y - x) - S_{2N-1}(y + x)}{2} \quad (1.4.5)$$

$$K_{USp(2N)}(x, y) = \frac{S_{2N+1}(y - x) - S_{2N+1}(y + x)}{2}. \quad (1.4.6)$$

The scaled limit of these kernel functions are given by

$$K_U(x, y) = S(y - x) \quad (1.4.7)$$

$$K_{Sp}(x, y) = S(y - x) - S(y + x) \quad (1.4.8)$$

$$K_O(x, y) = S(y - x) + S(y + x). \quad (1.4.9)$$

The  $n$ -level densities are given in the the following theorem

**Theorem 1.4.2.** *Let  $f$  be a rapidly decaying smooth test function and let  $\theta_1, \dots, \theta_n$  denote the eigenangles associated with the eigenvalues,  $e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_N}$  of  $A \in G(N) = U(N)$ ,  $USp(2N)$  or  $SO(2N)$ . Then we have:*

(i)

$$\begin{aligned} & \int_{U(N)} \sum_{1 \leq j_1 < \dots < j_n \leq N} f(\theta_{j_1}, \dots, \theta_{j_n}) dA \\ &= \frac{1}{n!(2\pi)^n} \int_{[0, 2\pi]^n} f(\theta_1, \dots, \theta_n) \det_{n \times n} S_N(\theta_k - \theta_j) d\theta_1 \dots d\theta_n. \end{aligned} \quad (1.4.10)$$

(ii)

$$\begin{aligned} & \int_{USp(2N)} \sum_{\substack{B \subset [1, N] \\ |B|=n}} f_B(\theta) dA \\ &= \frac{1}{n!(\pi)^n} \int_{[0, \pi]^n} f(\theta_1, \dots, \theta_n) \det_{n \times n} (K_{USp(2N)}(\theta_k, \theta_j)) d\theta_1 \dots d\theta_n. \end{aligned} \quad (1.4.11)$$

(iii)

$$\begin{aligned} & \int_{SO(2N)} \sum_{\substack{BC[1,N] \\ |B|=n}} f_B(\theta) dA \\ &= \frac{1}{n!(\pi)^n} \int_{[0,\pi]^n} f(\theta_1, \dots, \theta_n) \det_{n \times n} (K_{SO(2N)}(\theta_k, \theta_j)) d\theta_1 \dots d\theta_n. \end{aligned} \quad (1.4.12)$$

If we look for the normalized eigenangles, we have the following theorem

**Theorem 1.4.3.** *Using the same notation as before and calling  $\tilde{\theta}_j = \theta \frac{N}{2\pi}$  for  $U(N)$  and  $\hat{\theta}_j = \theta \frac{N}{\pi}$  for  $SO(2N)$  and  $USp(2N)$  to be the normalized eigenangles, we have that:*

(i)

$$\begin{aligned} & \lim_{N \rightarrow \infty} \int_{U(N)} \sum_{1 \leq j_1 < \dots < j_n \leq N} f(\tilde{\theta}_{j_1}, \dots, \tilde{\theta}_{j_n}) dA \\ &= \frac{1}{n!} \int_{\mathbb{R}_+^n} f(\tilde{\theta}_1, \dots, \tilde{\theta}_n) \det_{n \times n} S_N(\theta_k - \theta_j) d\theta_1 \dots d\theta_n. \end{aligned} \quad (1.4.13)$$

(ii)

$$\begin{aligned} & \lim_{N \rightarrow \infty} \int_{USp(2N)} \sum_{\substack{BC[1,N] \\ |B|=n}} f_B(\hat{\theta}) dA \\ &= \frac{1}{n!} \int_{\mathbb{R}_+^n} f(\theta_1, \dots, \theta_n) \det_{n \times n} (K_{USp}(\theta_k, \theta_j)) d\theta_1 \dots d\theta_n. \end{aligned} \quad (1.4.14)$$

(iii)

$$\begin{aligned} & \lim_{N \rightarrow \infty} \int_{SO(2N)} \sum_{\substack{BC[1,N] \\ |B|=n}} f_B(\hat{\theta}) dA \\ &= \frac{1}{n!} \int_{\mathbb{R}_+^n} f(\theta_1, \dots, \theta_n) \det_{n \times n} (K_{SO}(\theta_k, \theta_j)) d\theta_1 \dots d\theta_n. \end{aligned} \quad (1.4.15)$$

Instead of considering eigenvalues and eigenangles of matrices we can consider studying zeros of families of  $L$ -functions and we can define in an analogous way the  $n$ -level density of the zeros for these families and study how they are distributed within families. Studies of the  $n$ -level density for  $L$ -functions



were carried out by Iwaniec, Luo and Sarnak [Iwan-Luo-Sar], Hughes and Rudnick [Hug-Rud], Katz and Sarnak [Katz-Sar99a], Ozluk and Snyder [Ozl-Sny93, Ozl-Sny99], Peng Gao [Gao05, Gao08] and Rubinstein [Rub-ESIZ, Rub-LZR].

### 1.4.2 $k$ th–consecutive spacings

Let  $A \in G(N) = U(N)$ ,  $USp(2N)$  or  $SO(2N)$  jointly with the associated Haar measure  $dA$ , and let  $e^{i\theta_1(A)}, e^{i\theta_2(A)}, \dots, e^{i\theta_N(A)}$  be the eigenvalues of  $A$ . Katz–Sarnak defined the  $k$ th–consecutive spacing to be

$$\mu_k(A)[a, b] = \frac{\#\{1 \leq j \leq N : \frac{N}{2\pi}(\theta_{j+k} - \theta_j) \in [a, b]\}}{N}, \quad (1.4.16)$$

where

$$0 \leq \theta_1(A) \leq \theta_2(A) \leq \dots \leq \theta_N(A) < 2\pi. \quad (1.4.17)$$

For  $k = 1$  this statistics is called the *nearest neighbour spacing*. Katz and Sarnak showed that

**Theorem 1.4.4** (Katz-Sarnak). *The limit*

$$\lim_{N \rightarrow \infty} \int_{G(N)} \mu_k(A) dA \quad (1.4.18)$$

*exists and is the same for  $G(N) = U(N)$ ,  $USp(2N)$  and  $SO(2N)$ .*

### 1.4.3 $k$ th–lowest eigenvalue

Using the same notation of the previous subsection, Katz and Sarnak defined the distribution of the  $k$ th–lowest eigenvalue of  $A$ , when  $A$  varies over  $G(N)$  to be

$$\nu_k(G(N))[a, b] = \text{Haar} \left\{ A \in G(N) : \frac{\theta_k(A)N}{2\pi} \in [a, b] \right\}. \quad (1.4.19)$$

Next, Katz and Sarnak [Katz-Sar99a, Katz-Sar99b] showed that

**Theorem 1.4.5** (Katz–Sarnak). *There are measures  $\nu_k(G)$  which depends on the symmetry type of  $G$  such that*

$$\lim_{N \rightarrow \infty} \nu_k(G(N)) = \nu_k(G), \quad (1.4.20)$$

where  $\nu_k(G)$  can be expressed in terms of Fredholm determinants.

Under the Katz–Sarnak philosophy, Jon Keating and Nina Snaith were naturally led to raise the following question:

**Question 1.4.6.** *Is it possible to model (predict) the mean values (moments) of  $L$ -functions by using random matrix theory?*

Now we will discuss this question.

## 1.5 Characteristic Polynomials and the Keating–Snaith Conjectures

The analogies and similarities involving the statistics of eigenvalues of random matrices and zeros of the Riemann zeta–function and other  $L$ -functions were the motivations for the random matrix model introduced by Keating–Snaith [Kea-Sna00a, Kea-Sna00b], in which the Riemann zeta–function and other  $L$ -functions are modeled by characteristic polynomials of large random matrices.

The characteristic polynomial of an  $N \times N$  unitary matrix is given by

$$\Lambda_A(s) = \det(I - A^\dagger s) = \prod_{n=1}^N (1 - se^{-i\theta_n}), \quad (1.5.1)$$

where  $A^\dagger$  is the hermitian conjugate of  $A$ . And for an  $2N \times 2N$  unitary matrix with either  $A \in USp(2N)$  or  $A \in SO(2N)$ , we have the eigenvalues occur in conjugate pairs and thus the characteristic polynomial of such matrices can be written as

$$\Lambda_A(s) = \det(I - A^\dagger s) = \prod_{n=1}^N (1 - se^{i\theta_n})(1 - se^{-i\theta_n}). \quad (1.5.2)$$

If we scale the zeros of the Riemann zeta–function and the eigenphases of  $N \times N$  unitary matrices to have unit mean spacing, Keating and Snaith noted that was natural to equate the mean densities of each, that is

$$\frac{N}{2\pi} = \frac{1}{2\pi} \log \frac{T}{2\pi}, \quad (1.5.3)$$

and with this relation in hands they were able to establish some results and formulate some conjectures that we will now state. These results and conjectures can be seen as a true revolution in understanding the connection between Random Matrix Theory and the theory of the Riemann zeta–function and other  $L$ –functions.

**Theorem 1.5.1** (Keating–Snaith). *Let  $B \subset \mathbb{C}$  be a rectangle, then*

$$\lim_{N \rightarrow \infty} \text{Haar} \left\{ A \in U(N) : \frac{\log \Lambda_A(e^0)}{\sqrt{\frac{1}{2} \log N}} \in B \right\} = \frac{1}{2\pi} \int_B e^{-(x^2+y^2)/2} dx dy. \quad (1.5.4)$$

The above theorem is analogous to the following theorem proved by Selberg (see, for example [Lau]),

**Theorem 1.5.2** (Selberg). *Let  $B \subset \mathbb{C}$  be a rectangle, then*

$$\lim_{T \rightarrow \infty} \frac{1}{T} \left| \left\{ t : T \leq t \leq 2T : \frac{\log \zeta(\frac{1}{2} + it)}{\sqrt{\frac{1}{2} \log \log T}} \in B \right\} \right| = \frac{1}{2\pi} \int_B e^{-(x^2+y^2)/2} dx dy. \quad (1.5.5)$$

### 1.5.1 Moments of $L$ –functions and Characteristic Polynomials

This section is an attempt to answer the question raised in Section 1.4. In [Kea-Sna00a], Keating and Snaith established the following result using a form of Selberg’s integral

**Theorem 1.5.3** (Keating–Snaith). *For all  $\theta \in \mathbb{R}$  and  $\Re(s) > -1$  we have,*

$$\begin{aligned} M_N(s) &= \langle |\Lambda_A(e^{i\theta})|^s \rangle_{U(N)} = \int_{U(N)} |\Lambda_A(e^{i\theta})|^s dA \\ &= \frac{G^2(1 + \frac{1}{2}s)G(N+1)G(N+1+s)}{G(1+s)G^2(N+1 + \frac{1}{2}s)} = \prod_{j=1}^N \frac{\Gamma(j)\Gamma(j+s)}{\Gamma^2(j + \frac{1}{2}s)}, \end{aligned} \quad (1.5.6)$$

where  $G$  is the Barnes'  $G$ -function (see Appendix C) and the average is computed with respect to the Haar measure  $dA$  over  $U(N)$ .

As a corollary of the above theorem, we have

**Corollary 1.5.4.** *For integers  $k \geq 0$ ,*

$$M_N(2k) = \frac{G^2(k+1)}{G(2k+1)} N^{k^2} + O(N^{k^2-1}), \quad (1.5.7)$$

as  $N \rightarrow \infty$ , that is,  $M_N(2k)$  is the  $2k^{\text{th}}$  moment of  $\Lambda_A(e^0)$ , which is a polynomial in  $N$  of degree  $k^2$ .

Extending the above result to  $USp(2N)$  and  $SO(2N)$  Keating and Snaith obtained in [Kea-Sna00b] the following theorem

**Theorem 1.5.5** (Keating–Snaith).

$$\begin{aligned} \int_{U(N)} |\Lambda_A(e^{i\theta})|^{2k} dA &= \prod_{j=0}^{k-1} \left( \frac{j!}{(k+j)!} \prod_{i=1}^k (N+i+j) \right) \\ &\sim \left( \prod_{j=0}^{k-1} \frac{j!}{(k+j)!} \right) N^{k^2}, \end{aligned} \quad (1.5.8)$$

$$\begin{aligned} \int_{USp(2N)} |\Lambda_A(e^0)|^k dA &= 2^{k(k+1)/2} \left( \prod_{j=0}^k \frac{j!}{(2j)!} \right) \prod_{j=1}^k (N+j) \prod_{1 \leq i < j \leq k} (N + \frac{i+j}{2}) \\ &\sim 2^{k(k+1)/2} \left( \prod_{j=1}^k \frac{j!}{(2j)!} \right) N^{k(k+1)/2}, \end{aligned} \quad (1.5.9)$$

and

$$\begin{aligned} \int_{SO(2N)} |\Lambda_A(e^0)|^k dA &= 2^{k(k+1)/2} \left( \prod_{j=0}^{k-1} \frac{j!}{(2j)!} \right) \prod_{1 \leq i < j \leq k} (N-1 + \frac{i+j}{2}) \\ &\sim 2^{k(k+1)/2} \left( \prod_{j=1}^{k-1} \frac{j!}{(2j)!} \right) N^{k(k-1)/2}. \end{aligned} \quad (1.5.10)$$

Now, using the correspondence  $N \sim \log \frac{T}{2\pi}$ , Keating and Snaith conjectured the following formulas

**Conjecture 1.5.6** (Keating–Snaith). *For  $k$  fixed and  $\Re(k) > -\frac{1}{2}$ , we have*

$$\frac{1}{T} \int_0^T |\zeta(\frac{1}{2} + it)|^{2k} dt \sim a_k \frac{G^2(k+1)}{G(2k+1)} (\log T)^{k^2}, \quad (1.5.11)$$

as  $T \rightarrow \infty$  and

$$a_k = \prod_{p \text{ prime}} \left( \left(1 - \frac{1}{p}\right)^{k^2} \sum_{m=0}^{\infty} \frac{d_k(p^m)^2}{p^m} \right). \quad (1.5.12)$$

**Conjecture 1.5.7** (Keating–Snaith). *For  $k$  fixed and  $\Re(k) \geq 0$ , we have*

$$\frac{1}{\varphi^*(q)} \sum_{\chi(\bmod q)}^* |L(\frac{1}{2}, \chi)|^{2k} \sim a_k \frac{G^2(k+1)}{G(2k+1)} \prod_{p|q} \left( \sum_{m=0}^{\infty} \frac{d_k(p^m)^2}{p^m} \right)^{-1} (\log q)^{k^2}, \quad (1.5.13)$$

as  $q \rightarrow \infty$  and  $a_k$  is a product over primes.

And for the symplectic family, Keating and Snaith conjectured that

**Conjecture 1.5.8** (Keating–Snaith). *For  $k$  fixed and  $\Re(k) \geq 0$ , we have*

$$\frac{1}{D^*} \sum_{0 < d \leq D}^* L(\frac{1}{2}, \chi_{8d})^k \sim a_k \frac{G(k+1) \sqrt{\Gamma(k+1)}}{\sqrt{G(2k+1) \Gamma(2k+1)}} (\log D)^{k(k+1)/2}, \quad (1.5.14)$$

as  $D \rightarrow \infty$  and

$$a_k = 2^{-k(k+2)/2} \prod_{p \geq 3} \frac{(1 - \frac{1}{p})^{k(k+1)/2}}{1 + \frac{1}{p}} \left( \frac{(1 - \frac{1}{\sqrt{p}})^{-k} + (1 + \frac{1}{\sqrt{p}})^{-k}}{2} + \frac{1}{p} \right). \quad (1.5.15)$$

These conjectures for moments of  $L$ -functions in unitary, symplectic and also in orthogonal families (not listed here) are discussed in greater detail in [CFKRS05] where it is possible to include lower order terms in the asymptotic series for the moments. We will discuss the conjectures presented in [CFKRS05] in much more detail in Chapter 6.

## 1.6 Overview of this Thesis

This thesis splits into two parts. The first part consist of Chapter 3, Chapter 4 and Chapter 5, where we establish some mean value theorems for  $L$ -functions and for the class number in the context of function fields using the new analytic techniques developed by Faifman & Rudnick [Fai-Rud] and Kurlberg & Rudnick [Kur-Rud] used to study similar problems. The second part consist of Chapter 6, Chapter 7 and Chapter 8, where we extend to the function field setting the heuristic previously developed, by Conrey *et.al.* [CFKRS05, Conr-Far-Zir], for the integral moments and ratios of  $L$ -functions defined over number fields. Specifically, we give a heuristic for the moments and ratios of a family of  $L$ -functions associated with hyperelliptic curves of genus  $g$  over a fixed finite field  $\mathbb{F}_q$  in the limit as  $g \rightarrow \infty$ . As an application, we calculate the one-level density for the zeros of these  $L$ -functions.

# Chapter 2

## Function Field Preliminaries

In this chapter we will give some background on Number Theory over Function Fields. We will use the Rosen's book [Rosen] as a general reference. We also suggest to the reader the books by Goss [Goss] and Thakur [Thakur] who deal with the function field arithmetic and for the study of additive number theory of polynomials over finite fields we recommend the book by Effinger and Hayes [Eff-Hay].

### 2.1 Polynomials over Finite Fields

In this section we present some known facts about finite fields and polynomials over finite fields.

#### 2.1.1 Basic facts about $\mathbb{F}_q$ and $\mathbb{F}_q[T]$

Let  $\mathbb{F}_q$  denote a finite field with  $q$  elements. The general model for such a field is  $\mathbb{Z}/p\mathbb{Z}$ , where  $p$  is a prime number. We have that  $\mathbb{Z}/p\mathbb{Z}$  is a finite field and has  $p$  elements and in general the number of elements in a finite field is a power of a prime number,  $q = p^y$ . In this case the number  $p$  is called the characteristic of  $\mathbb{F}_q$ .

We will denote by  $A = \mathbb{F}_q[T]$  the polynomial ring over  $\mathbb{F}_q$ . It is known

that  $A$  has many properties in common with the ring of integers  $\mathbb{Z}$ . For a detailed discussion of the similarities between  $A$  and  $\mathbb{Z}$  see [Rosen, Chapter 1] and Carlitz's original paper [Carlitz]. Let  $f(T) \in A$ , so

$$f(T) = a_n T^n + a_{n-1} T^{n-1} + \cdots + a_1 T + a_0. \quad (2.1.1)$$

If  $a_n \neq 0$  we say that  $f$  has degree  $n$ , i.e.,  $\deg(f) = n$  and in this case we define the *sign* of  $f$  to be  $a_n \in \mathbb{F}_q^*$ , ( $\text{sgn}(f) = a_n$ ). We have that  $\text{sgn}(0) = 0$  and  $\deg(0) = -\infty$ . We now present some simple properties involving  $f$

**Proposition 2.1.1.** *Let  $f, g \in A$  be non-zero polynomials. Then,*

(i)  $\deg(fg) = \deg(f) + \deg(g)$ ,

(ii)  $\text{sgn}(fg) = \text{sgn}(f)\text{sgn}(g)$ ,

(iii)  $\deg(f + g) \leq \max(\deg(f), \deg(g))$  and equality holds if  $\deg(f) \neq \deg(g)$ .

A polynomial  $f \in A$  is called *monic* if  $\text{sgn}(f) = 1$ . And a polynomial  $f \in A$  is *reducible* if we can write  $f(T) = a(T)b(T)$  with  $\deg(a) > 0$  and  $\deg(b) > 0$ , otherwise is called *irreducible* (see, for example [Ire-Ros, Chapter 1]). We also have the following important definition associated with  $f \in A$

**Definition 2.1.2.** *The norm of a polynomial  $f \in \mathbb{F}_q[T]$  is defined in the following way. For  $f \neq 0$ , set  $|f| := q^{\deg f}$  and if  $f = 0$ , set  $|f| = 0$ .*

A monic irreducible polynomial is called a “prime” polynomial. We have that  $A$  has the unique factorization property, that is, every  $f \in A$ ,  $f \neq 0$ , can be written uniquely in the form

$$f = \alpha P_1^{e_1} P_2^{e_2} \cdots P_t^{e_t}, \quad (2.1.2)$$

where  $\alpha \in \mathbb{F}_q^*$  and each  $P_i$  is a monic irreducible polynomial,  $P_i \neq P_j$  for  $i \neq j$  and  $e_i$  is a non-negative integer for  $i = 1, 2, \dots, t$ .

**Definition 2.1.3.** *The zeta function of  $A = \mathbb{F}_q[T]$ , denoted by  $\zeta_A(s)$ , is defined by the infinite series*

$$\zeta_A(s) := \sum_{\substack{f \in A \\ f \text{ monic}}} \frac{1}{|f|^s} = \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} (1 - |P|^{-s})^{-1}, \quad \Re(s) > 1. \quad (2.1.3)$$



And is easy to show that

$$\zeta_A(s) = \frac{1}{1 - q^{1-s}}. \quad (2.1.4)$$

We define the gamma function of  $A$ , denoted by  $\Gamma_A(s)$ , by

$$\Gamma_A(s) := \frac{1}{(1 - q^{-s})}. \quad (2.1.5)$$

And we have the following theorem

**Theorem 2.1.4.** *The zeta-function  $\zeta_A(s)$  can be continued to a meromorphic function to the whole complex plane with a simple pole at  $s = 1$  with residue  $1/\log q$ . If we define  $\xi_A(s) = q^{-s}\Gamma_A(s)\zeta_A(s)$ , then*

$$\xi_A(s) = \xi_A(1 - s). \quad (2.1.6)$$

*Proof.* From (2.1.4) we have that  $\zeta_A(s)$  can be continued to a meromorphic function. A simple calculation shows that the residue at  $s = 1$  is  $1/\log q$ . And (2.1.6) follows directly from the definition of  $\zeta_A(s)$  and  $\Gamma_A(s)$ .  $\square$

We can also define the analogue of the Möbius function, denoted by  $\mu(f)$  in this case. And the Euler totient function, denoted by  $\Phi(f)$  for  $A = \mathbb{F}_q[T]$  as follows

$$\mu(f) = \begin{cases} (-1)^t, & f = \alpha P_1 P_2 \dots P_t, \\ 0, & \text{otherwise,} \end{cases} \quad (2.1.7)$$

where each  $P_j$  is a distinct monic irreducible polynomial and

$$\Phi(f) = \sum_{\substack{g \text{ monic} \\ \deg(g) < \deg(f) \\ (f,g)=1}} 1. \quad (2.1.8)$$

### 2.1.2 Prime Number Theorem in $A = \mathbb{F}_q[T]$

We now present the analogue of the Prime Number Theorem for polynomials over finite fields.

**Theorem 2.1.5** (Prime Number Theorem for Polynomials). *Let  $\pi_A(n)$  denote the number of monic irreducible polynomials in  $A = \mathbb{F}_q[T]$  of degree  $n$ . Then,*

$$\pi_A(n) = \frac{q^n}{n} + O\left(\frac{q^{n/2}}{n}\right). \quad (2.1.9)$$

**Remark 2.1.6.** *If we denote  $x = q^n$ , we have*

$$\pi_A(n) = \frac{x}{\log_q x} + O\left(\frac{\sqrt{x}}{\log_q x}\right), \quad (2.1.10)$$

*which looks like the conjectured precise form of the classical prime number theorem.*

## 2.2 Dirichlet $L$ -functions over the rational function field $\mathbb{F}_q(T)$

In this section we will discuss the basic properties of  $L$ -functions over the rational function field  $\mathbb{F}_q(T)$ .

### 2.2.1 Characters and the Reciprocity Law

Assume that  $q$  is odd and let  $P(x) \in \mathbb{F}_q[T]$  be an irreducible polynomial. Then by [Rosen, Proposition 1.10], if  $f \in A$  and  $P \nmid f$  we know that the congruence  $x^d \equiv f \pmod{P}$  is solvable if and only if

$$f^{\frac{|P|-1}{d}} \equiv 1 \pmod{P},$$

where  $d$  is a divisor of  $q - 1$ . So if  $P \nmid f$ , let  $(f/P)_d$  be the unique element of  $\mathbb{F}_q^*$  such that

$$f^{\frac{|P|-1}{d}} \equiv \left(\frac{f}{P}\right)_d \pmod{P}$$

and if  $P \mid f$  we define  $(f/P)_d = 0$ .

Now we can define the quadratic residue symbol  $(f/P) \in \{\pm 1\}$  for  $f$  coprime to  $P$  by

$$\left(\frac{f}{P}\right) \equiv f^{(|P|-1)/2} \pmod{P}. \quad (2.2.1)$$

We can then define the Jacobi symbol  $(f/Q)$  for arbitrary monic  $Q$ . Let  $f$  be coprime to  $Q$  and  $Q = P_1^{e_1} P_2^{e_2} \dots P_s^{e_s}$  so

$$\left(\frac{f}{Q}\right) = \prod_{j=1}^s \left(\frac{f}{P_j}\right)^{e_j}.$$

If  $f, Q$  are not coprime we set  $(f/Q) = 0$  and if  $\alpha \in \mathbb{F}_q^*$  is a scalar then

$$\left(\frac{\alpha}{Q}\right) = \alpha^{((q-1)/2)\deg Q}. \quad (2.2.2)$$

The analogue of the quadratic reciprocity law for  $A = \mathbb{F}_q[T]$  is

**Theorem 2.2.1** (Quadratic reciprocity). *Let  $A, B \in \mathbb{F}_q[T]$  be relatively prime polynomials and  $A \neq 0$  and  $B \neq 0$ . Then,*

$$\left(\frac{A}{B}\right) = \left(\frac{B}{A}\right) (-1)^{((q-1)/2)\deg(A)\deg(B)} = \left(\frac{B}{A}\right) (-1)^{(|A|-1)(|B|-1)/2}. \quad (2.2.3)$$

## 2.2.2 General Dirichlet $L$ -functions

**Definition 2.2.2.** *Let  $Q \in A = \mathbb{F}_q[T]$  be a monic polynomial. A Dirichlet character modulo  $Q$  is defined to be a function  $\chi : A \rightarrow \mathbb{C}$  which satisfies the following properties:*

- (i)  $\chi(f + gQ) = \chi(f)$  for all  $f, g \in A$ ,
- (ii)  $\chi(f)\chi(g) = \chi(fg)$  for all  $f, g \in A$ ,
- (iii)  $\chi(f) \neq 0$  if and only if  $(f, Q) = 1$ .

We have that the trivial character  $\chi_0(f)$  is defined to be  $\chi_0(f) = 1$  if  $(f, Q) = 1$  and  $\chi_0(f) = 0$  otherwise. The Dirichlet character modulo  $Q$  is a homomorphism from  $(A/QA)^* \rightarrow \mathbb{C}^*$  and the number of Dirichlet characters modulo  $Q$  is given by  $\Phi(Q) = \#((A/QA)^*)$ . We have the following result for Dirichlet characters modulo  $Q$

**Proposition 2.2.3** (Orthogonality Relations). *Let  $\chi$  and  $\psi$  two Dirichlet characters modulo  $Q$  and let  $f$  and  $g$  be elements of  $A$  which are relatively prime to  $Q$ . Then,*

(i)

$$\sum_f \chi(f) \overline{\psi(f)} = \Phi(Q) \delta(\chi, \psi), \quad (2.2.4)$$

(ii)

$$\sum_{\chi} \chi(f) \overline{\chi(g)} = \Phi(Q) \delta(f, g), \quad (2.2.5)$$

where the first sum is over any set of representative for  $A/QA$  and the second sum is over all Dirichlet characters modulo  $Q$ . And  $\delta(\chi, \psi) = 0$  if  $\chi \neq \psi$  and  $\delta(\chi, \psi) = 1$  if  $\chi = \psi$  and similarly for  $\delta(f, g)$ .

**Definition 2.2.4.** Let  $\chi$  be a Dirichlet character modulo  $Q \in A = \mathbb{F}_q[T]$ . The Dirichlet  $L$ -function associated with  $\chi$  is defined by

$$L(s, \chi) = \sum_{f \text{ monic}} \frac{\chi(f)}{|f|^s}, \quad \Re(s) > 1. \quad (2.2.6)$$

We have in this way that  $L(s, \chi)$  converges absolutely for  $\Re(s) > 1$  and since the characters are multiplicative we can deduce that

$$L(s, \chi) = \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left(1 - \frac{\chi(P)}{|P|^s}\right)^{-1}, \quad \Re(s) > 1. \quad (2.2.7)$$

We can write the  $L$ -function associated with the trivial character as,

$$L(s, \chi_0) = \prod_{\substack{P \text{ monic} \\ \text{irreducible} \\ P|Q}} \left(1 - \frac{1}{|P|^s}\right) \zeta_A(s). \quad (2.2.8)$$

This shows that  $L(s, \chi_0)$  can be analytically continued to the whole complex plane and has a simple pole at  $s = 1$ .

We now present an important result concerning  $L$ -functions in function fields that make the theory in this case very attractive,

**Theorem 2.2.5.** Let  $\chi$  be a non-trivial Dirichlet character modulo  $Q$ . Then,  $L(s, \chi)$  is a polynomial in  $u = q^{-s}$  of degree at most  $\deg(Q) - 1$ .

And we have the following corollary from the theorem

**Corollary 2.2.6.** If  $\chi$  is non-trivial, then  $L(s, \chi)$  can be analytically continued to an entire function for the whole complex plane  $\mathbb{C}$ .

At this point we can make a distinction between characters. For any monic polynomial  $Q$ , we call a character of  $(A/QA)^*$  *even* if  $\chi(\alpha) = 1$  for all  $\alpha \in \mathbb{F}_q^*$ . Otherwise,  $\chi$  is said to be an *odd* character. Proceeding in the same way as done in the classical case, we have that  $L(s, \chi)$  also satisfies a functional equation (see [Weil-BNT])

**Proposition 2.2.7.** *Let  $\chi$  be an even Dirichlet character modulo  $Q$ . If we denote*

$$\xi(s, \chi) = \frac{q^{\frac{s(-2+\deg(Q))}{2}}}{1 - q^{-s}} L(s, \chi).$$

Then,

$$\xi(s, \chi) = c_\chi \xi(1 - s, \bar{\chi}), \quad |c_\chi| = 1. \quad (2.2.9)$$

### 2.2.3 Quadratic Characters and the Corresponding $L$ -functions

**Definition 2.2.8.** *Let  $D \in \mathbb{F}_q[T]$  be a square-free polynomial. We define the quadratic character  $\chi_D$  using the quadratic residue symbol for  $\mathbb{F}_q[T]$  by*

$$\chi_D(f) = \left( \frac{D}{f} \right). \quad (2.2.10)$$

So, if  $P \in A$  is a monic irreducible polynomial we have

$$\chi_D(P) = \begin{cases} 0, & \text{if } P \mid D, \\ 1, & \text{if } P \nmid D \text{ and } D \text{ is a square modulo } P, \\ -1, & \text{if } P \nmid D \text{ and } D \text{ is a non square modulo } P. \end{cases}$$

We define the  $L$ -function corresponding to the quadratic character  $\chi_D$  by

$$\mathcal{L}(u, \chi_D) := \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} (1 - \chi_D(P)u^{\deg P})^{-1}, \quad |u| < 1/q \quad (2.2.11)$$

where  $u = q^{-s}$ . The  $L$ -function above can also be expressed as an infinite series in the usual way

$$\mathcal{L}(u, \chi_D) = \sum_{\substack{f \in A \\ f \text{ monic}}} \chi_D(f)u^{\deg f} = L(s, \chi_D) = \sum_{\substack{f \in A \\ f \text{ monic}}} \frac{\chi_D(f)}{|f|^s}. \quad (2.2.12)$$

We can write (2.2.12) as

$$\mathcal{L}(u, \chi_D) = \sum_{n \geq 0} \sum_{\substack{\deg(f)=n \\ f \text{ monic}}} \chi_D(f) u^n, \quad (2.2.13)$$

and if we denote

$$A_D(n) := \sum_{\substack{f \text{ monic} \\ \deg(f)=n}} \chi_D(f),$$

we can write (2.2.13) as

$$\sum_{n \geq 0} A_D(n) u^n. \quad (2.2.14)$$

By Theorem 2.2.5, if  $D$  is a non-square polynomial of positive degree, then  $A_D(n) = 0$  for  $n \geq \deg(D)$  and in this case the  $L$ -function is in fact a polynomial of degree at most  $\deg(D) - 1$ .

We will now assume the primitivity condition that  $D$  is a square-free monic polynomial of positive degree. Following the arguments presented in [Rud-TPFHE] we have that  $\mathcal{L}(u, \chi_D)$  has a “trivial” zero at  $u = 1$  if and only if  $\deg(D)$  is even, which enable us to define the “completed”  $L$ -function

$$\mathcal{L}(u, \chi_D) = (1 - u)^\lambda \mathcal{L}^*(u, \chi_D), \quad \lambda = \begin{cases} 1, & \deg(D) \text{ even,} \\ 0, & \deg(D) \text{ odd,} \end{cases} \quad (2.2.15)$$

where  $\mathcal{L}^*(u, \chi_D)$  is a polynomial of even degree

$$2\delta = \deg(D) - 1 - \lambda$$

satisfying the functional equation

$$\mathcal{L}^*(u, \chi_D) = (qu^2)^\delta \mathcal{L}^*\left(\frac{1}{qu}, \chi_D\right). \quad (2.2.16)$$

## 2.2.4 Zeta functions associated with Curves

Let  $\mathbb{F}_q$  be a fixed finite field of odd cardinality and  $A = \mathbb{F}_q[T]$  be the polynomial ring over  $\mathbb{F}_q$  in the variable  $T$  as earlier. Let  $C$  be any smooth, projective, geometrically connected curve of genus  $g \geq 1$  defined over the finite field  $\mathbb{F}_q$ .

The zeta function of the curve  $C$  was introduced by Artin [Artin] and is defined as

$$Z_C(u) := \exp \left( \sum_{n=1}^{\infty} N_n(C) \frac{u^n}{n} \right), \quad |u| < 1/q \quad (2.2.17)$$

where  $N_n(C) := \text{Card}(C(\mathbb{F}_{q^n}))$  is the number of points on  $C$  with coordinates in a field extension  $\mathbb{F}_{q^n}$  of  $\mathbb{F}_q$  of degree  $n \geq 1$ . The Weil conjectures states that (see [Weil-CA, Weil-CAD])

**Theorem 2.2.9** (The Weil Conjectures). *Let  $C$  be a given curve of genus  $g$  as above. Then we have,*

(i)[Rationality]  $Z_C(u)$  is a rational function of  $u = q^{-s}$ . More precisely,

$$Z_C(u) = \zeta_C(s) = \frac{P_C(q^{-s})}{(1 - q^{-s})(1 - q^{1-s})}, \quad (2.2.18)$$

where  $P_C(u) \in \mathbb{Z}[u]$  is a polynomial of degree  $2g$  called of  $L$ -polynomial of the curve  $C$ , with  $P_C(0) = 1$ .

(ii)[Functional Equation] Denote  $\xi_C(s) = q^{(g-1)s} \zeta_C(s)$ . Then for all  $s$  one has

$$\xi_C(s) = \xi_C(1 - s). \quad (2.2.19)$$

For the polynomial  $P_C(u)$  the functional equation translates to

$$P_C(u) = (qu^2)^g P_C\left(\frac{1}{qu}\right). \quad (2.2.20)$$

(iii) [Riemann Hypothesis] All the roots of  $\zeta_C(s)$  lie on the line  $\Re(s) = 1/2$ . Equivalently, the inverse roots of  $P_C(u)$  all have absolute value  $\sqrt{q}$ .

By the Riemann Hypothesis for curves over finite fields, proved by Weil [Weil-CAD], one knows that all zeros of  $P_C(u)$  lie on the circle  $|u| = q^{-1/2}$ , i.e.,

$$P_C(u) = \prod_{j=1}^{2g} (1 - \alpha_j u), \quad \text{with } |\alpha_j| = \sqrt{q} \text{ for all } j. \quad (2.2.21)$$

By [Rosen, Proposition 14.6 and 17.7], the  $L$ -function  $\mathcal{L}^*(u, \chi_D)$  is the Artin  $L$ -function attached to the unique nontrivial quadratic character of

$\mathbb{F}_q(T)(\sqrt{D(T)})$ . An extremely important fact for this thesis is that the numerator  $P_C(u)$  of the zeta-function associated with the hyperelliptic curve  $C_D : y^2 = D(T)$  coincides with the completed Dirichlet  $L$ -function  $\mathcal{L}^*(u, \chi_D)$  associated with the quadratic character  $\chi_D$  as was found in Artin's thesis. So we can write  $\mathcal{L}^*(u, \chi_D)$  as

$$\mathcal{L}^*(u, \chi_D) = \sum_{n=0}^{2\delta} A_D^*(n)u^n, \quad (2.2.22)$$

where  $A_D^*(0) = 1$  and  $A_D^*(2\delta) = q^\delta$ .

If  $D$  is a monic and square-free polynomial of positive degree, the zeta function (2.2.18) of the hyperelliptic curve  $C_D : y^2 = D(T)$  can be written as

$$Z_{C_D}(u) = \frac{\mathcal{L}^*(u, \chi_D)}{(1-u)(1-qu)}. \quad (2.2.23)$$

### 2.2.5 Spectral Interpretation

The Riemann Hypothesis for curves over finite fields, proved by Weil [Weil-CAD], shows that all zeros of  $Z_C(u)$  and hence of  $P_C(u)$ , lie on the circle  $|u| = q^{-1/2}$ .

So the polynomial  $P_C(u)$  is the characteristic polynomial of an unitary symplectic matrix  $\Theta_C \in USp(2g)$ , defined up to conjugacy, and we can write

$$P_C(u) = \det(I - u\sqrt{q}\Theta_{C_D}). \quad (2.2.24)$$

The eigenvalues of  $\Theta_C$  are of the form  $e(\theta_{C,j})$ ,  $j = 1, \dots, 2g$ , where  $e(\theta) = e^{2\pi i\theta}$ .

For a fixed genus  $g$ , Katz and Sarnak [Katz-Sar99a] showed that the conjugacy classes (Frobenius classes)  $\{\Theta_C : C \in \mathcal{H}\}$ , where  $\mathcal{H}$  is an appropriate family of curves, becomes equidistributed in the unitary symplectic group  $USp(2g)$  (with respect to Haar measure) in the limit  $q \rightarrow \infty$ . That is, for any continuous function  $F$  on the space of conjugacy classes of  $USp(2g)$ ,

$$\lim_{q \rightarrow \infty} \frac{1}{\#\mathcal{H}} \sum_{C \in \mathcal{H}} F(\Theta_C) = \int_{USp(2g)} F(A) dA, \quad (2.2.25)$$



where  $dA$  is the Haar measure and the sum indicates that  $C$  varies over a family of curves  $\mathcal{H}$ .

This result allows one to compute arithmetic quantities such as moments of  $\log(P_C(u))$  and the moments of  $P_C(u)$  when  $C$  varies over a family of curves  $\mathcal{H}$  by using the corresponding computation from Random Matrix Theory for  $USp(2g)$ . For example, let  $\mathcal{M}_g$  be the family of all  $k$ -isomorphism classes of (smooth, projective and geometrically connected) curves of genus  $g$  (see, [Katz-Sar99a, Theorem 10.7.15] and [Kea-Lin-Rud, Sections 4.1 and 4.2]) and for  $u = q^{-1/2}$ , the critical point in this case, one has

$$\lim_{q \rightarrow \infty} \frac{1}{\#\mathcal{M}_g(k)} \sum_{C \in \mathcal{M}_g(k)} (P_C(q^{-1/2}))^s = \int_{USp(2g)} \det(I - A)^s dA. \quad (2.2.26)$$

Keating and Snaith [Kea-Sna00b] computed the moments of the characteristic polynomial in  $USp(2g)$  were they found the formula given in Theorem 1.5.5.

As said earlier the goal of this thesis is to explore the opposite limit,  $q$  fixed and  $g \rightarrow \infty$ . In this case the matrices  $\Theta_C$  inhabit different spaces as  $g$  grows, and we do not know how to formulate an equidistribution problem à la Deligne for this case. For a description of these types of problems, the Katz–Sarnak philosophy and the interaction of random matrices with function fields see the excellent survey by Douglas Ulmer [Ulmer].

The investigation of problems involving the limit  $g \rightarrow \infty$  and  $q$  fixed were initiated by Faifman [Faifman] and Faifman–Rudnick [Fai-Rud] in the study on the distribution of zeros of the zeta functions of hyperelliptic curves and by Kurlberg–Rudnick [Kur-Rud] in the study of the fluctuations in the number of points on a hyperelliptic curve. Rudnick [Rud-TPFHE] has studied the mean value of traces of high powers of the Frobenius class as  $g \rightarrow \infty$  and Bucur–David *et.al.* [BDFL] studied the variation of the trace of the Frobenius endomorphism in the cyclic trigonal ensemble extending the results of Kurlberg and Rudnick.

## 2.2.6 The Hyperelliptic Ensemble $\mathcal{H}_{2g+1,q}$

Let  $\mathcal{H}_d$  be the set of square-free monic polynomials of degree  $d$  in  $\mathbb{F}_q[T]$ . The cardinality of  $\mathcal{H}_d$  is

$$\#\mathcal{H}_d = \begin{cases} (1 - 1/q)q^d, & d \geq 2, \\ q, & d = 1, \end{cases}$$

to count the number of square-free monic polynomials of prescribed degree we use that

$$\sum_{d>0} \frac{\#\mathcal{H}_d}{q^{ds}} = \sum_{\substack{f \text{ monic} \\ \text{squarefree}}} |f|^{-s} = \frac{\zeta_A(s)}{\zeta_A(2s)}$$

and equation (2.1.4) to obtain the above result, the full prove of this can be found in [Rosen, Proposition 2.3]. In particular for  $g \geq 1$  we have that,

$$\#\mathcal{H}_{2g+1,q} = (q-1)q^{2g} = \frac{|D|}{\zeta_A(2)}. \quad (2.2.27)$$

We can see  $\mathcal{H}_{2g+1,q}$  as a probability space (ensemble) with the uniform probability measure (for more details see Appendix B). Thus the expected value of any continuous function  $F$  on  $\mathcal{H}_{2g+1,q}$  is defined as

$$\langle F(D) \rangle := \frac{1}{\#\mathcal{H}_{2g+1,q}} \sum_{D \in \mathcal{H}_{2g+1,q}} F(D). \quad (2.2.28)$$

Using the Möbius function  $\mu$  of  $\mathbb{F}_q[T]$  defined in (2.1.7) we can sieve out the square-free polynomials as is done over the integers via

$$\sum_{A^2|D} \mu(A) = \begin{cases} 1, & D \text{ square-free,} \\ 0, & \text{otherwise.} \end{cases} \quad (2.2.29)$$

And in this way we can write the expected value of any function  $F$  as

$$\begin{aligned} \langle F(D) \rangle &= \frac{1}{\#\mathcal{H}_{2g+1,q}} \sum_{\substack{D \text{ monic} \\ \deg(D)=2g+1}} \sum_{A^2|D} \mu(A) F(D) \\ &= \frac{1}{(q-1)q^{2g}} \sum_{2\alpha+\beta=2g+1} \sum_{\substack{B \text{ monic} \\ \deg B=\beta}} \sum_{\substack{A \text{ monic} \\ \deg A=\alpha}} \mu(A) F(A^2B). \end{aligned} \quad (2.2.30)$$

## 2.3 Average Value Theorems of $L$ -functions in Function Fields

**Theorem 2.3.1** (Hoffstein–Rosen [Hoff-Ros]). *Let  $M$  be odd and positive. We have that for  $s \neq \frac{1}{2}$  the following formula holds*

$$q^{-M} \sum_{\substack{m \text{ monic} \\ \deg(m)=M}} L(s, \chi_m) = \frac{\zeta_A(2s)}{\zeta_A(2s+1)} - \left(1 - \frac{1}{q}\right) (q^{1-2s})^{\frac{M+1}{2}} \zeta_A(2s). \quad (2.3.1)$$

For  $s = \frac{1}{2}$ , we have

$$q^{-M} \sum_{\substack{m \text{ monic} \\ \deg(m)=M}} L\left(\frac{1}{2}, \chi_m\right) = 1 + \left(1 - \frac{1}{q}\right) \left(\frac{M-1}{2}\right). \quad (2.3.2)$$

Now we state the next mean value theorem

**Theorem 2.3.2** (Hoffstein–Rosen [Hoff-Ros]). *Let  $M$  be even and positive. The following sums are over all non-square monic polynomials of degree  $M$ . For  $s \neq \frac{1}{2}$  or 1. Then we have,*

$$q^{-M} \sum L(s, \chi_m) = \frac{\zeta_A(2s)}{\zeta_A(2s+1)} - \left(1 - \frac{1}{q}\right) (q^{1-2s})^{\frac{M}{2}} \zeta_A(2s) - q^{-\frac{M}{2}} \left( \frac{\zeta_A(2s)}{\zeta_A(2s+1)} - \left(1 - \frac{1}{q}\right) (q^{1-s})^M \zeta_A(s) \right). \quad (2.3.3)$$

For  $s = 1$  we have,

$$q^{-M} \sum L(1, \chi_m) = \frac{\zeta_A(2)}{\zeta_A(3)} - q^{-\frac{M}{2}} \left( 2 + \left(1 - \frac{1}{q}\right) (M-1) \right). \quad (2.3.4)$$

Now quoting Rosen (Chapter 17 [Rosen]) ‘...a more difficult problem is to consider only polynomials  $D$  that are square-free. In this case,  $\mathcal{O}_D$  is the integral closure of  $A = \mathbb{F}_q[T]$  in  $K = k(\sqrt{D})$ . In the language of binary quadratic forms, we would be restricting the average by consider forms with fundamental discriminants. Averaging in this case is surprisingly difficult’. But Hoffstein and Rosen succeeded in this task by obtaining the following result

**Theorem 2.3.3** (Hoffstein–Rosen [Hoff-Ros]). *Let*

$$P(s) = \prod_P \left( 1 - \frac{1}{|P|^2} - \frac{1}{|P|^{4s}} + \frac{1}{|P|^{4s+1}} \right). \quad (2.3.5)$$

*Then for any  $\varepsilon > 0$  we have*

$$\sum_{\substack{\deg(m)=2n+1 \\ m \text{ square-free}}} L\left(\frac{1}{2}, \chi_m\right) = q^{2n+2} P\left(\frac{1}{2}\right) (d_3 + (1 - q^{-1})(n + 1)) + O(q^n(1 + \varepsilon)), \quad (2.3.6)$$

*where the  $d_3$  is the constant term in a certain Laurent expansion.*

In a recent paper A. Bucur and A. Diaconu [Buc-Diac] established the following result

**Theorem 2.3.4** (Bucur–Diaconu). *As  $q \rightarrow \infty$ , we have*

$$\sum_{\substack{d \in A \\ d \text{ monic} \\ \deg(d)=2g}} L\left(\frac{1}{2}, \chi_d\right)^4 \sim \frac{g(1+g)^2(2+g)^2(3+g)(1+2g)(3+2g)^2(5+2g)}{75600} q^{2g}. \quad (2.3.7)$$

Note that Theorem 2.3.4 is the fourth power moment for Quadratic Dirichlet  $L$ -Functions as  $q \rightarrow \infty$  and  $g$  is fixed. Therefore, this is the opposite limit discussed in this thesis and thus we are led to think that the right thing to do is to consider and impose the condition that  $d$  is square-free and then invoke the equidistribution results in Katz–Sarnak [Katz-Sar99a, Theorem 9.2.6] to deduce that all moments of  $L(\frac{1}{2}, \chi_d)$  are given by RMT calculations. However, the result of Bucur and Diaconu (2.3.7), is much deeper, since it can be written as

$$\sum_{\substack{d \in A \\ d \text{ monic} \\ \deg(d)=2g}} L\left(\frac{1}{2}, \chi_d\right)^4 = \frac{1}{2\pi i} \int_{|y|=q^{-3}} \frac{Z_{\text{even}}(y) dy}{y^{g+1}}, \quad (2.3.8)$$

where  $Z_{\text{even}}(y)$  has the Laurent expansion

$$Z_{\text{even}}(y) = \frac{C_{-11}}{(1 - q^2 y)^{11}} + \cdots + \frac{C_1}{(1 - q^2 y)} + C_0 + \cdots \quad (2.3.9)$$

with  $C_{-j} = g_j + O(q^{-1/2})$  and

$$g_1, \dots, g_{11} := 0, 0, 0, -1, 38, -394, 1765, -4032, 4928, -3072, 768. \quad (2.3.10)$$

So they are able to show that,

$$\sum_{\substack{d \in A \\ d \text{ monic} \\ \deg(d)=2g}} L\left(\frac{1}{2}, \chi_d\right)^4 = q^{2g} \sum_{j=1}^{11} \binom{g+j-1}{g} C_{-j} + \frac{1}{2\pi i} \int_{|y|=q^{-8/5}} \frac{Z_{\text{even}}(y) dy}{y^{g+1}}, \quad (2.3.11)$$

where the integral can be bounded.

To conclude, they gave a formula without the need to take the  $g$  or  $q$  limits and from their formula for the fourth power moment is possible to take the  $q$  or  $g$  limit at the end, after all calculations. Therefore, can be the case that instead take the  $q$  limit as above we are interested in take the  $g$  limit and for this situation we cannot invoke the powerful Katz–Sarnak equidistribution results, but the formula presented by Bucur and Diaconu allow us to obtain the asymptotic for the fourth moment in the  $g$  limit (note that the  $g$  limit is not presented in their paper). To establish the result above Bucur and Diaconu make use of the Multiple Dirichlet Series machinery and the Weyl group action of a particular Kac-Moody algebra to handle the infinite group of functional equations.

# Chapter 3

## The Mean Value of $L(\frac{1}{2}, \chi)$ in the Hyperelliptic Ensemble

In this chapter we obtain an asymptotic formula for the first moment of quadratic Dirichlet  $L$ -functions over the rational function field at the central point  $s = \frac{1}{2}$ . Specifically, we compute the expected value of  $L(\frac{1}{2}, \chi_D)$  for the ensemble  $\mathcal{H}_{2g+1, q}$  of hyperelliptic curves of genus  $g$  over a fixed finite field as  $g \rightarrow \infty$ . Our approach relies on the use of the analogue of the approximate functional equation for such  $L$ -functions and our formula includes the main lower order terms. The results presented in this chapter are the function field analogues of those obtained previously by Jutila [Jutila] in the number-field setting and are consistent with recent general conjectures for the moments of  $L$ -functions motivated by Random Matrix Theory. The main theorem presented in this chapter also appears in the paper by Andrade and Keating [And-Kea11].

In other words, we prove in this chapter the function field analogue of the following theorem

**Theorem 3.0.5** (Jutila). *Let  $L(s, \chi_d)$  be the quadratic Dirichlet  $L$ -function associated with the quadratic character  $\chi_d(n)$  as defined in Section 1.2.3. Then*

we have,

$$\begin{aligned} & \sum_{0 < d \leq D} L\left(\frac{1}{2}, \chi_d\right) \\ &= \frac{P(1)}{4\zeta(2)} D \left\{ \log(D/\pi) + \frac{\Gamma'}{\Gamma}(1/4) + 4\gamma - 1 + 4\frac{P'}{P}(1) \right\} + O(D^{3/4+\varepsilon}), \end{aligned} \quad (3.0.1)$$

where

$$P(s) = \prod_p \left( 1 - \frac{1}{(p+1)p^s} \right),$$

and the sum is taken over fundamental discriminants  $d$ .

Recall from Chapter 2 that quadratic Dirichlet  $L$ -functions for function fields consist of those  $L$ -functions with Dirichlet series and Euler product

$$L(s, \chi_D) = \sum_{f \text{ monic}} \frac{\chi_D(f)}{|f|^s} \quad \text{and} \quad L(s, \chi_D) = \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left( 1 - \frac{\chi_D(P)}{|P|^s} \right)^{-1} \quad (3.0.2)$$

respectively, where  $\chi_D(f)$  is the quadratic character defined by the Jacobi Symbol  $\chi_D(f) = \left(\frac{D}{f}\right)$ , and for our purpose  $D \in \mathcal{H}_{2g+1,q}$ .

Further, remember that the  $L$ -function associated with the hyperelliptic curve  $C_D : y^2 = D(T)$ , where  $D \in \mathcal{H}_{2g+1,q}$  satisfies the functional equation given by

$$L(s, \chi_D) = |D|^{\frac{1}{2}-s} \mathcal{X}(s) L(1-s, \chi_D), \quad (3.0.3)$$

where

$$\mathcal{X}(s) = q^{-\frac{1}{2}+s}. \quad (3.0.4)$$

## 3.1 Statement of the Theorem

Based on the works of Conrey and Soundararajan [Conr-Sound], Jutila [Jutila] and Matthew Young [Young-FMQDL] we prove the following theorem which can be seen as the function field analogue of the Jutila's result (3.0.1).

**Theorem 3.1.1.** *Let  $q$  be the fixed cardinality of the ground field  $\mathbb{F}_q$  and assume for simplicity that  $q \equiv 1 \pmod{4}$ . Then*

$$\begin{aligned} & \sum_{D \in \mathcal{H}_{2g+1, q}} L\left(\frac{1}{2}, \chi_D\right) \\ &= \frac{P(1)}{2\zeta_A(2)} |D| \left\{ \log_q |D| + 1 + \frac{4}{\log q} \frac{P'}{P}(1) \right\} + O\left(|D|^{3/4 + \frac{\log_q 2}{2}}\right), \end{aligned} \quad (3.1.1)$$

where

$$P(s) = \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left( 1 - \frac{1}{(|P|+1)|P|^s} \right), \quad (3.1.2)$$

$|f| = q^{\deg(f)}$  for any polynomial  $f \in \mathbb{F}_q[T]$  (so  $|D| = q^{2g+1}$ ), and

$$\zeta_A(s) = \frac{1}{1 - q^{1-s}} \quad (3.1.3)$$

is the zeta function associated to  $A = \mathbb{F}_q[T]$ .

A direct corollary of the Theorem 3.1.1 using (2.2.27) and computing the limit as  $g \rightarrow \infty$  is

**Corollary 3.1.2.** *Under the same assumptions of Theorem 3.1.1 we have that,*

$$\frac{1}{\#\mathcal{H}_{2g+1, q}} \sum_{D \in \mathcal{H}_{2g+1, q}} L\left(\frac{1}{2}, \chi_D\right) \sim \frac{1}{2} P(1) (\log_q |D|) = \frac{1}{2} P(1) (2g+1) \quad (3.1.4)$$

as  $g \rightarrow \infty$ .

**Corollary 3.1.3.** *From Theorem 3.1.1 we have that,*

$$L\left(\frac{1}{2}, \chi_D\right) \neq 0, \quad (3.1.5)$$

for infinitely many square-free monic polynomials  $D \in A$  of degree  $2g+1$ .

Previously, Hoffstein and Rosen [Hoff-Ros] obtained an asymptotic formula for the first moment of Dirichlet  $L$ -functions over function fields (see Theorem 2.3.3) making use of Eisenstein series for the metaplectic two-fold cover of  $GL(2, k_\infty)$ , where  $k_\infty$  is the completion of  $k = \mathbb{F}_q(T)$  at the prime at infinity.

One important difference between Hoffstein and Rosen's result and Theorem 3.1.1 is that we sum over square-free and monic polynomials, which



means that we are averaging over positive and fundamental discriminants in this setting. The two results have the same general form, but are different in their details. Our calculation is complementary to that developed in [Hoff-Ros], being more similar to the classical methods employed in [Jutila] using the analytic techniques developed by [Fai-Rud], [Kur-Rud] and [Rud-TPFHE] to deal with the function field case.

## 3.2 “Approximate” Functional Equation

The starting point in the proof of Theorem 3.1.1 is a representation for  $L(s, \chi_D)$ , which can be viewed as the analogue of the approximate functional equation for the Riemann zeta function (equation 4.12.4 in [Tit]) or for the quadratic Dirichlet  $L$ -function (Lemma 3 in [Jutila]). In our case the formula is an identity rather than an approximation.

**Lemma 3.2.1** (“Approximate” Functional Equation). *Let  $\chi_D$  be a quadratic character, where  $D \in \mathcal{H}_{2g+1, q}$ . Then*

$$\begin{aligned} \mathcal{L}^*(q^{-1/2}, \chi_D) &= \mathcal{L}(q^{-1/2}, \chi_D) \\ &= \sum_{n=0}^g \sum_{\substack{f_1 \text{ monic} \\ \deg(f_1)=n}} \chi_D(f_1) q^{-n/2} + \sum_{m=0}^{g-1} \sum_{\substack{f_2 \text{ monic} \\ \deg(f_2)=m}} \chi_D(f_2) q^{-m/2}. \end{aligned} \quad (3.2.1)$$

*Proof.* Using the ideas presented by Conrey *et.al.* in [CFKRS05] we substitute  $\mathcal{L}^*(u, \chi_D) = \sum_{n=0}^{2g} a_n u^n$  into the functional equation (2.2.20)

$$\begin{aligned} \sum_{n=0}^{2g} a_n u^n &= q^g u^{2g} \sum_{m=0}^{2g} a_m \left(\frac{1}{qu}\right)^m = q^g u^{2g} \sum_{m=0}^{2g} a_m q^{-m} u^{-m} \\ &= \sum_{m=0}^{2g} a_m q^{g-m} u^{2g-m} = \sum_{k=0}^{2g} a_{2g-k} q^{k-g} u^k. \end{aligned}$$

Therefore,

$$\sum_{n=0}^{2g} a_n u^n = \sum_{k=0}^{2g} a_{2g-k} q^{k-g} u^k.$$

Equating coefficients we have that

$$a_n = a_{2g-n}q^{n-g} \quad \text{or} \quad a_{2g-n} = a_nq^{g-n},$$

and so we can write the polynomial  $\mathcal{L}^*(u, \chi_D)$  as

$$\begin{aligned} \sum_{n=0}^{2g} a_n u^n &= \sum_{n=0}^g a_n u^n + \sum_{m=0}^{g-1} a_{2g-m} u^{2g-m} \\ &= \sum_{n=0}^g a_n u^n + \sum_{m=0}^{g-1} a_m q^{g-m} u^{2g-m} \\ &= \sum_{n=0}^g a_n u^n + q^g u^{2g} \sum_{m=0}^{g-1} a_m q^{-m} u^{-m}. \end{aligned} \quad (3.2.2)$$

Writing  $a_n = \sum_{\substack{f \text{ monic} \\ \deg(f)=n}} \chi_D(f)$  and  $u = q^{-1/2}$  in (3.2.2) proves the lemma.  $\square$

We can write the polynomial  $\mathcal{L}^*(u, \chi_D)$  using the variable  $s$  and so (3.2.2) becomes

$$\mathcal{L}(u, \chi_D) = L(s, \chi_D) = \sum_{\substack{f_1 \text{ monic} \\ \deg(f_1) \leq g}} \frac{\chi_D(f_1)}{|f_1|^s} + (q^{1-2s})^g \sum_{\substack{f_2 \text{ monic} \\ \deg(f_2) \leq g-1}} \frac{\chi_D(f_2)}{|f_2|^{1-s}}. \quad (3.2.3)$$

### 3.3 Setting Up The Problem

The basic quantity of study can be viewed from (2.2.22) and (2.2.23), as being

$$\sum_{D \in \mathcal{H}_{2g+1, q}} \mathcal{L}(q^{-1/2}, \chi_D) = \sum_{D \in \mathcal{H}_{2g+1, q}} \sum_{n=0}^{2g} \sum_{\substack{f \text{ monic} \\ \deg(f)=n}} \chi_D(f) q^{-n/2}. \quad (3.3.1)$$

Using the Lemma 3.2.1 we can save  $g$  terms and write (3.3.1) as

$$\begin{aligned} &\sum_{D \in \mathcal{H}_{2g+1, q}} \mathcal{L}(q^{-1/2}, \chi_D) \\ &= \sum_{D \in \mathcal{H}_{2g+1, q}} \sum_{n=0}^g \sum_{\substack{f_1 \text{ monic} \\ \deg(f_1)=n}} \frac{\chi_D(f_1)}{q^{n/2}} + \sum_{D \in \mathcal{H}_{2g+1, q}} \sum_{m=0}^{g-1} \sum_{\substack{f_2 \text{ monic} \\ \deg(f_2)=m}} \frac{\chi_D(f_2)}{q^{m/2}}. \end{aligned} \quad (3.3.2)$$

As both terms on the right-hand side of (3.3.2) are similar we need only worry about computing one of them to obtain the final result.

### 3.3.1 Averaging the Approximate Functional Equation

We are interested in obtaining an asymptotic formula for the first term on the RHS of (3.3.2) and so we need to compute

$$\begin{aligned}
 & \sum_{D \in \mathcal{H}_{2g+1,q}} \sum_{n=0}^g \sum_{\substack{f \text{ monic} \\ \deg(f)=n}} \chi_D(f) q^{-n/2} \\
 &= \sum_{n=0}^g q^{-n/2} \sum_{D \in \mathcal{H}_{2g+1,q}} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f=\square=l^2}} \chi_D(f) + \sum_{n=0}^g q^{-n/2} \sum_{D \in \mathcal{H}_{2g+1,q}} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f \neq \square}} \chi_D(f) \\
 &= \sum_{\substack{n=0 \\ 2|n}}^g q^{-n/2} \sum_{\substack{l \text{ monic} \\ \deg(l)=n/2}} \sum_{D \in \mathcal{H}_{2g+1,q}} \chi_D(l^2) + \sum_{n=0}^g q^{-n/2} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f \neq \square}} \sum_{D \in \mathcal{H}_{2g+1,q}} \chi_D(f) \\
 &= \sum_{\substack{n=0 \\ 2|n}}^g q^{-n/2} \sum_{\substack{l \text{ monic} \\ \deg(l)=n/2}} \sum_{\substack{D \in \mathcal{H}_{2g+1,q} \\ (D,l)=1}} 1 + \sum_{n=0}^g q^{-n/2} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f \neq \square}} \sum_{D \in \mathcal{H}_{2g+1,q}} \chi_D(f), \quad (3.3.3)
 \end{aligned}$$

where the first term on the RHS of the final expression corresponds to contributions of squares to the average and the second term to the non-square contributions.

Basically, the problem is the following: for the square contributions we need to count square-free polynomials which are coprime to a fixed monic polynomial and to perform the summation over monic polynomials  $l$  and over integers  $n$  up to  $g$ , and for the non-square contributions the difficulty is to average the non-trivial quadratic character.

## 3.4 The Main Term Calculation

In this section we will derive an asymptotic formula for

$$\sum_{\substack{n=0 \\ 2|n}}^g q^{-n/2} \sum_{\substack{l \text{ monic} \\ \deg(l)=n/2}} \sum_{\substack{D \in \mathcal{H}_{2g+1,q} \\ (D,l)=1}} 1, \quad (3.4.1)$$

which corresponds to the contributions of squares to the average. As in the number field case, the contribution of squares gives the main term of the first

moment. The principal result in this section is

**Proposition 3.4.1.** *With the same notation as in Theorem 3.1.1,*

$$\sum_{\substack{n=0 \\ 2|n}}^g q^{-n/2} \sum_{\substack{l \text{ monic} \\ \deg(l)=n/2}} \sum_{\substack{D \in \mathcal{H}_{2g+1,q} \\ (D,l)=1}} 1 = \frac{P(1)}{\zeta_A(2)} |D| \left\{ ([g/2] + 1) + \sum_{\substack{P \text{ monic} \\ \text{irreducible}}} \frac{\deg P}{|P|(|P| + 1) - 1} \right\} + O(gq^{\frac{3}{2}g}). \quad (3.4.2)$$

We will need some preliminary lemmas.

### 3.4.1 Sieving out Square-Free and Coprime Polynomials.

In this section we will need from following proposition presented in Rosen [Rosen, Proposition 2.4]

**Proposition 3.4.2.** *We have that,*

$$\Phi(f) = |f| \prod_{P|f} (1 - |P|^{-1}). \quad (3.4.3)$$

We will prove here the following proposition

**Proposition 3.4.3.**

$$\sum_{\substack{D \in \mathcal{H}_{2g+1,q} \\ (D,l)=1}} 1 = \frac{|D|}{\zeta_A(2) \prod_{P|l} (1 + |P|^{-1})} + O\left(\sqrt{|D|} \frac{\Phi(l)}{|l|}\right). \quad (3.4.4)$$

We will need the following lemmas

**Lemma 3.4.4.** *Let  $V_d = \{D \in \mathbb{F}_q[T] : D \text{ monic, } \deg(D) = d\}$ . Then,*

$$\#\{D \in V_d : (D, l) = 1\} = q^d \frac{\Phi(l)}{|l|}. \quad (3.4.5)$$

*Proof.*

$$\begin{aligned}
 \#\{D \in V_d : (D, l) = 1\} &= \sum_{\substack{D \text{ monic} \\ \deg(D)=d \\ (D, l)=1}} 1 = \sum_{\substack{D \text{ monic} \\ \deg(D)=d}} \sum_{h|(D, l)} \mu(h) \\
 &= \sum_{h|l} \mu(h) \sum_{\substack{D \text{ monic} \\ \deg(D)=d \\ h|D}} 1 = \sum_{h|l} \mu(h) \sum_{\substack{m \text{ monic} \\ \deg(m)=d-\deg h}} 1 \\
 &= \sum_{h|l} \mu(h) q^{d-\deg(h)} = q^d \sum_{h|l} \frac{\mu(h)}{|h|} \\
 &= q^d \prod_{\substack{P \\ P|l}} \left(1 - \frac{1}{|P|}\right) = q^d \frac{\Phi(l)}{|l|} \tag{3.4.6}
 \end{aligned}$$

where we used Proposition 3.4.2 in (3.4.6).  $\square$

**Lemma 3.4.5.** *We have that,*

$$\sum_{\substack{Q \text{ monic} \\ \deg(Q) > \frac{2g+1}{2} \\ (Q, l)=1}} \frac{\mu(Q)}{|Q|^2} \ll q^{-1/2} q^{-g}. \tag{3.4.7}$$

*Proof.*

$$\begin{aligned}
 \sum_{\substack{Q \text{ monic} \\ \deg(Q) > \frac{2g+1}{2} \\ (Q, l)=1}} \frac{\mu(Q)}{|Q|^2} &\leq \sum_{\substack{Q \text{ monic} \\ \deg(Q) > \frac{2g+1}{2} \\ (Q, l)=1}} \frac{1}{|Q|^2} \\
 &= \sum_{n > \frac{2g+1}{2}} \sum_{\substack{Q \text{ monic} \\ \deg(Q)=n}} \frac{1}{|Q|^2} \\
 &= \sum_{n > \frac{2g+1}{2}} \frac{1}{q^n} \ll q^{-1/2} q^{-g}. \tag{3.4.8}
 \end{aligned}$$

$\square$

**Lemma 3.4.6.** *We have that,*

$$\sum_{\substack{Q \text{ monic} \\ \deg(Q) \leq \frac{2g+1}{2} \\ (Q, l)=1}} \frac{\mu(Q)}{|Q|^2} = \frac{1}{\zeta_A(2)} \frac{1}{\prod_{P|l} (1 - 1/|P|^2)} + O(q^{-1/2} q^{-g}). \tag{3.4.9}$$

*Proof.*

$$\begin{aligned} \sum_{\substack{Q \text{ monic} \\ \deg(Q) \leq \frac{2g+1}{2} \\ (Q,l)=1}} \frac{\mu(Q)}{|Q|^2} &= \sum_{\substack{Q \text{ monic} \\ (Q,l)=1}} \frac{\mu(Q)}{|Q|^2} - \sum_{\substack{Q \text{ monic} \\ \deg(Q) > \frac{2g+1}{2} \\ (Q,l)=1}} \frac{\mu(Q)}{|Q|^2} \\ &= \prod_{P|l} \left(1 - \frac{1}{|P|^2}\right) - \sum_{\substack{Q \text{ monic} \\ \deg(Q) > \frac{2g+1}{2} \\ (Q,l)=1}} \frac{\mu(Q)}{|Q|^2}, \end{aligned} \quad (3.4.10)$$

and

$$\begin{aligned} \prod_{P|l} \left(1 - \frac{1}{|P|^2}\right) &= \prod_P \left(1 - \frac{1}{|P|^2}\right) \prod_{P|l} \left(1 - \frac{1}{|P|^2}\right)^{-1} \\ &= \frac{1}{\zeta_A(2)} \frac{1}{\prod_{P|l} (1 - 1/|P|^2)}. \end{aligned} \quad (3.4.11)$$

Therefore,

$$\sum_{\substack{Q \text{ monic} \\ \deg(Q) \leq \frac{2g+1}{2} \\ (Q,l)=1}} \frac{\mu(Q)}{|Q|^2} = \frac{1}{\zeta_A(2)} \frac{1}{\prod_{P|l} (1 - 1/|P|^2)} - \sum_{\substack{Q \text{ monic} \\ \deg(Q) > \frac{2g+1}{2} \\ (Q,l)=1}} \frac{\mu(Q)}{|Q|^2}, \quad (3.4.12)$$

and using the estimate of Lemma 3.4.5 proves the result.  $\square$

*Proof of Proposition 3.4.3.* Following the ideas presented in the proof of Lemma 4.2 in [BDFL] we have that

$$\begin{aligned} \sum_{\substack{D \in \mathcal{H}_{2g+1,q} \\ (D,l)=1}} 1 &= \sum_{\substack{D \in V_{2g+1} \\ (D,l)=1}} \sum_{Q^2|D} \mu(Q) = \sum_{\substack{Q \text{ monic} \\ \deg(Q) \leq \frac{2g+1}{2} \\ (Q,l)=1}} \mu(Q) \sum_{\substack{D \in V_{2g+1-2\deg(Q)} \\ (D,l)=1}} 1 \\ &= \sum_{\substack{Q \text{ monic} \\ \deg(Q) \leq \frac{2g+1}{2} \\ (Q,l)=1}} \mu(Q) \#\{D \in V_{2g+1-2\deg(Q)} : (D,l) = 1\}. \end{aligned} \quad (3.4.13)$$

By Lemma 3.4.4, we have

$$\begin{aligned} \sum_{\substack{D \in \mathcal{H}_{2g+1,q} \\ (D,l)=1}} 1 &= \sum_{\substack{Q \text{ monic} \\ \deg(Q) \leq \frac{2g+1}{2} \\ (Q,l)=1}} \mu(Q) q^{2g+1-2\deg(Q)} \frac{\Phi(l)}{|l|} \\ &= |D| \frac{\Phi(l)}{|l|} \sum_{\substack{Q \text{ monic} \\ \deg(Q) \leq \frac{2g+1}{2} \\ (Q,l)=1}} \frac{\mu(Q)}{|Q|^2}. \end{aligned} \quad (3.4.14)$$

Invoking Lemma 3.4.6 we obtain,

$$\begin{aligned} \sum_{\substack{D \in \mathcal{H}_{2g+1,q} \\ (D,l)=1}} 1 &= |D| \frac{\Phi(l)}{|l|} \left( \frac{1}{\zeta_A(2)} \frac{1}{\prod_{P|l} (1 - 1/|P|^2)} + O(q^{-1/2}q^{-g}) \right) \\ &= |D| \frac{\Phi(l)}{|l|} \frac{1}{\zeta_A(2)} \frac{1}{\prod_{P|l} (1 - 1/|P|^2)} + O\left(|D| \frac{\Phi(l)}{|l|} q^{-1/2}q^{-g}\right), \end{aligned}$$

and using  $\frac{\Phi(l)}{|l|} = \prod_{P|l} (1 - |P|^{-1})$ , we end up with

$$\sum_{\substack{D \in \mathcal{H}_{2g+1,q} \\ (D,l)=1}} 1 = \frac{|D|}{\zeta_A(2) \prod_{P|l} (1 + |P|^{-1})} + O\left(\sqrt{|D|} \frac{\Phi(l)}{|l|}\right), \quad (3.4.15)$$

which proves Proposition 3.4.3.  $\square$

### 3.4.2 A Sum Over Monic Polynomials.

In this section we establish the following two results:

**Lemma 3.4.7.** *We have that,*

$$\prod_{\substack{P \\ P|l}} (1 + |P|^{-1})^{-1} = \sum_{\substack{d \text{ monic} \\ d|l}} \mu(d) \prod_{P|d} \frac{1}{|P| + 1}. \quad (3.4.16)$$

*Proof.* Obviously,

$$\prod_{\substack{P \\ P|l}} (1 + |P|^{-1})^{-1} = \prod_{\substack{P \\ P|l}} \left(1 - \frac{1}{|P| + 1}\right).$$

Let  $P_1, \dots, P_m$  be the primes that divide  $l$ . Then

$$\begin{aligned} \prod_{\substack{P \\ P|l}} \left(1 - \frac{1}{|P| + 1}\right) &= \left(1 - \frac{1}{|P_1| + 1}\right) \left(1 - \frac{1}{|P_2| + 1}\right) \cdots \left(1 - \frac{1}{|P_m| + 1}\right) \\ &= 1 - \left(\frac{1}{|P_1| + 1} + \cdots + \frac{1}{|P_m| + 1}\right) \\ &\quad + \left(\frac{1}{|P_1| + 1} \frac{1}{|P_2| + 1} + \cdots\right) - \cdots \\ &= \sum_{\substack{d \text{ monic} \\ d|l}} \mu(d) \prod_{P|d} \frac{1}{|P| + 1}, \end{aligned} \quad (3.4.17)$$

which proves the lemma.  $\square$

**Lemma 3.4.8.** *We have that,*

$$\sum_{\substack{l \text{ monic} \\ \deg(l)=n/2}} \prod_{P|l} (1 + |P|^{-1})^{-1} = q^{n/2} \sum_{\substack{d \text{ monic} \\ \deg(d) \leq n/2}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1}. \quad (3.4.18)$$

*Proof.* Using Lemma 3.4.7 we have,

$$\begin{aligned} & \sum_{\substack{l \text{ monic} \\ \deg(l)=n/2}} \prod_{P|l} (1 + |P|^{-1})^{-1} = \sum_{\substack{l \text{ monic} \\ \deg(l)=n/2}} \sum_{\substack{d \text{ monic} \\ d|l}} \mu(d) \prod_{P|d} \frac{1}{|P|+1} \\ &= \sum_{\substack{d \text{ monic} \\ \deg(d) \leq n/2}} \sum_{\substack{l \text{ monic} \\ \deg(l)=n/2 \\ d|l}} \mu(d) \prod_{P|d} \frac{1}{|P|+1} = \sum_{\substack{d \text{ monic} \\ \deg(d) \leq n/2}} \mu(d) \prod_{P|d} \frac{1}{|P|+1} \sum_{\substack{l \text{ monic} \\ \deg(l)=n/2 \\ d|l}} 1 \\ &= \sum_{\substack{d \text{ monic} \\ \deg(d) \leq n/2}} \mu(d) \prod_{P|d} \frac{1}{|P|+1} q^{n/2 - \deg(d)} = q^{n/2} \sum_{\substack{d \text{ monic} \\ \deg(d) \leq n/2}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1}. \end{aligned}$$

□

### 3.4.3 Auxiliary Lemmas

To prove Proposition 3.4.1, which is the main result of this section, we will need some additional lemmas, the first one is quoted from Rosen [Rosen, Proposition 2.7]

**Lemma 3.4.9.** *We have that,*

$$\sum_{\substack{\deg(f)=n \\ f \text{ monic}}} \Phi(f) = q^{2n}(1 - q^{-1}). \quad (3.4.19)$$

**Lemma 3.4.10.** *We have that,*

$$\sum_{\substack{n=0 \\ 2|n}}^g q^{-n/2} \sum_{\substack{l \text{ monic} \\ \deg(l)=n/2}} \sqrt{|Q|} \frac{\Phi(l)}{|l|} = \sqrt{|Q|}(1 - q^{-1})([g/2] + 1). \quad (3.4.20)$$

*Proof.*

$$\begin{aligned} \sum_{\substack{n=0 \\ 2|n}}^g q^{-n/2} \sum_{\substack{l \text{ monic} \\ \deg(l)=n/2}} \sqrt{|Q|} \frac{\Phi(l)}{|l|} &= \sqrt{|Q|} \sum_{\substack{n=0 \\ 2|n}}^g q^{-n} \sum_{\substack{l \text{ monic} \\ \deg(l)=n/2}} \Phi(l) \\ &= \sqrt{|Q|} \sum_{\substack{n=0 \\ 2|n}}^g (1 - q^{-1}), \end{aligned}$$



where we have used Lemma 3.4.9 to obtain the last equation. Hence

$$\sqrt{|Q|} \sum_{\substack{n=0 \\ 2|n}}^g (1 - q^{-1}) = \sqrt{|Q|} (1 - q^{-1}) \sum_{\substack{n=0 \\ 2|n}}^g 1,$$

which proves the lemma, since  $n$  is even.  $\square$

So, from this lemma, we can conclude that

$$\sum_{\substack{n=0 \\ 2|n}}^g q^{-n/2} \sum_{\substack{l \text{ monic} \\ \deg(l)=n/2}} \sqrt{|Q|} \frac{\Phi(l)}{|l|} = O(gq^g), \quad (3.4.21)$$

which is a result that will be of use later.

Using the Euler product formula we can prove the following lemma

**Lemma 3.4.11.** *We have that,*

$$\sum_{d \text{ monic}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P| + 1} = \prod_P \left( 1 - \frac{1}{|P|(|P| + 1)} \right). \quad (3.4.22)$$

There are two additional lemmas which will be important in establishing the formula in Proposition 3.4.1.

**Lemma 3.4.12.** *We have that,*

$$([g/2] + 1) \sum_{\substack{d \text{ monic} \\ \deg(d) > [g/2]}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P| + 1} = O(gq^{-g/2}). \quad (3.4.23)$$

*Proof.*

$$\begin{aligned} \sum_{\substack{d \text{ monic} \\ \deg(d) > [g/2]}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P| + 1} &\leq \sum_{\substack{d \text{ monic} \\ \deg(d) > [g/2]}} \frac{\mu^2(d)}{|d|} \prod_{P|d} \frac{1}{|P|} \\ &\leq \sum_{\substack{d \text{ monic} \\ \deg(d) > [g/2]}} |d|^{-2} = \sum_{h > [g/2]} |d|^{-2} \sum_{\substack{d \text{ monic} \\ \deg(d)=h}} 1 \\ &= \sum_{h > [g/2]} q^{-h} \ll q^{-[g/2]} \ll q^{-g/2}. \end{aligned} \quad (3.4.24)$$

So,

$$([g/2] + 1) \sum_{\substack{d \text{ monic} \\ \deg(d) > [g/2]}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P| + 1} \ll gq^{-g/2}.$$

$\square$

**Lemma 3.4.13.** *We have that,*

$$\sum_{\substack{d \text{ monic} \\ \deg(d) > [g/2]}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} \deg(d) = O(gq^{-g/2}). \quad (3.4.25)$$

*Proof.* Using the same reasoning as in Lemma 3.4.12

$$\begin{aligned} & \sum_{\substack{d \text{ monic} \\ \deg(d) > [g/2]}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} \deg(d) \\ & \leq \sum_{\substack{d \text{ monic} \\ \deg(d) > [g/2]}} \frac{\mu^2(d)}{|d|} \prod_{P|d} \frac{1}{|P|} \deg(d) = \sum_{\substack{d \text{ monic} \\ \deg(d) > [g/2]}} |d|^{-2} \deg(d) \\ & = \sum_{h > [g/2]} \sum_{\substack{d \text{ monic} \\ \deg(d)=h}} hq^{-2h} = \sum_{h > [g/2]} hq^{-h} \ll [g/2]q^{-[g/2]} \ll gq^{-g/2}. \end{aligned}$$

Where we have used the following formula given in [Gra-Ryz]

$$\sum_{h=j+1}^{\infty} hq^{-h} = \frac{q^{-1-j}(-jq + q^2 + jq^2)}{(-1+q)^2}. \quad (3.4.26)$$

□

Next, we will establish the following formula

**Proposition 3.4.14.** *We have that,*

$$\begin{aligned} & \sum_{d \text{ monic}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} \deg(d) \\ & = - \prod_P \left( 1 - \frac{1}{|P|(|P|+1)} \right) \sum_{\substack{P \text{ monic} \\ \text{irreducible}}} \frac{\deg(P)}{|P|(|P|+1)-1}. \end{aligned} \quad (3.4.27)$$

*Proof.* Let,

$$f(s) = \sum_{d \text{ monic}} \deg(d) \frac{\mu(d)}{|d|^s} \prod_{P|d} \frac{1}{|P|+1} \quad (3.4.28)$$

and

$$g(s) = \sum_{d \text{ monic}} \frac{\mu(d)}{|d|^s} \prod_{P|d} \frac{1}{|P|+1}. \quad (3.4.29)$$

A simple calculation shows that

$$g'(s) = -f(s) \log q \quad (3.4.30)$$

and by Lemma 3.4.11

$$g(s) = \prod_P \left( 1 - \frac{1}{|P|^s(|P| + 1)} \right). \quad (3.4.31)$$

Computing  $g'(s)$  using (3.4.31) and the product rule gives us

$$g'(s) = g(s) \log q \sum_{\substack{P \text{ monic} \\ \text{irreducible}}} \frac{\deg(P)}{|P|^s(|P| + 1) - 1}. \quad (3.4.32)$$

Combining (3.4.30) and (3.4.32) we have that

$$f(s) = -g(s) \sum_{\substack{P \text{ monic} \\ \text{irreducible}}} \frac{\deg(P)}{|P|^s(|P| + 1) - 1}. \quad (3.4.33)$$

Putting  $s = 1$  in the last formula, proves the theorem.  $\square$

Now we are ready to give a proof of our main result of this section.

*Proof of Proposition 3.4.1.* Let

$$B(n, l, D) = \sum_{\substack{n=0 \\ 2|n}}^g q^{-n/2} \sum_{\substack{l \text{ monic} \\ \deg(l)=n/2}} \sum_{\substack{D \in \mathcal{H}_{2g+1, q} \\ (D, l)=1}} 1. \quad (3.4.34)$$

By Proposition 3.4.3 we have that

$$\begin{aligned} B(n, l, D) &= \frac{|D|}{\zeta_A(2)} \sum_{\substack{n=0 \\ 2|n}}^g q^{-n/2} \sum_{\substack{l \text{ monic} \\ \deg(l)=n/2}} \prod_{P|l} (1 + |P|^{-1})^{-1} \\ &+ O \left( \sum_{\substack{n=0 \\ 2|n}}^g q^{-n/2} \sum_{\substack{l \text{ monic} \\ \deg(l)=n/2}} \sqrt{|D|} \frac{\Phi(l)}{|l|} \right) \end{aligned}$$

and using (3.4.21) we can reduce  $B(n, l, D)$  to

$$B(n, l, D) = \frac{|D|}{\zeta_A(2)} \sum_{\substack{n=0 \\ 2|n}}^g q^{-n/2} \sum_{\substack{l \text{ monic} \\ \deg(l)=n/2}} \prod_{P|l} (1 + |P|^{-1})^{-1} + O(gq^g). \quad (3.4.35)$$

Using Lemma 3.4.8 we have that

$$\begin{aligned}
 B(n, l, D) &= \frac{|D|}{\zeta_A(2)} \sum_{\substack{n=0 \\ 2|n}}^g q^{-n/2} q^{n/2} \sum_{\substack{d \text{ monic} \\ \deg(d) \leq n/2}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} + O(gq^g) \\
 &= \frac{|D|}{\zeta_A(2)} \sum_{m=0}^{[g/2]} \sum_{\substack{d \text{ monic} \\ \deg(d) \leq m}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} + O(gq^g) \\
 &= \frac{|D|}{\zeta_A(2)} \sum_{\substack{d \text{ monic} \\ \deg(d) \leq m}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} \sum_{\deg(d) \leq m \leq [g/2]} 1 + O(gq^g) \\
 &= \frac{|D|}{\zeta_A(2)} \sum_{\substack{d \text{ monic} \\ \deg(d) \leq [g/2]}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} ([g/2] + 1 - \deg(d)) + O(gq^g).
 \end{aligned} \tag{3.4.36}$$

Hence

$$\begin{aligned}
 B(n, l, D) &= \frac{|D|}{\zeta_A(2)} \left\{ ([g/2] + 1) \left( \sum_{d \text{ monic}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} \right) \right\} \\
 &\quad - \frac{|D|}{\zeta_A(2)} \left\{ ([g/2] + 1) \left( \sum_{\substack{d \text{ monic} \\ \deg(d) > [g/2]}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} \right) \right\} \\
 &\quad - \frac{|D|}{\zeta_A(2)} \left\{ \sum_{d \text{ monic}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} \deg(d) \right\} \\
 &\quad + \frac{|D|}{\zeta_A(2)} \left\{ \sum_{\substack{d \text{ monic} \\ \deg(d) > [g/2]}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} \deg(d) \right\} + O(gq^g).
 \end{aligned} \tag{3.4.37}$$

The main term comes from the two sums over all monic polynomials. The sums over monic polynomials with  $\deg(d) > [g/2]$  can be bounded. So, we can write  $B(n, l, D)$  as

$$B(n, l, D) = (\mathbf{Main-Term}) + (\mathbf{Error-Term}) + O(gq^g). \tag{3.4.38}$$

Combining Lemma 3.4.11, Lemma 3.4.12, Lemma 3.4.13 and Proposition 3.4.14

we have,

$$\begin{aligned}
 B(n, l, D) &= \frac{|D|}{\zeta_A(2)} ([g/2] + 1) P(1) + \frac{|D|}{\zeta_A(2)} P(1) \sum_{\substack{P \text{ monic} \\ \text{irreducible}}} \frac{\deg(P)}{|P|(|P| + 1) - 1} \\
 &\quad + O(gq^{\frac{3}{2}g}) \\
 &= \frac{P(1)}{\zeta_A(2)} |D| \left\{ ([g/2] + 1) + \sum_{\substack{P \text{ monic} \\ \text{irreducible}}} \frac{\deg(P)}{|P|(|P| + 1) - 1} \right\} + O(gq^{\frac{3}{2}g}),
 \end{aligned}$$

which completes the proof of the proposition.  $\square$

### 3.5 Estimating the Contributions of Non-Squares to the Average.

We will present in this section an estimate for the second term of (3.3.3) which allows us to give an asymptotic formula for the first term of (3.3.2) where  $q \equiv 1 \pmod{4}$  is fixed and  $g \rightarrow \infty$ . Our main result in this section is

**Proposition 3.5.1.** *We have that,*

$$\sum_{n=0}^g q^{-n/2} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f \neq \square}} \sum_{D \in \mathcal{H}_{2g+1, q}} \chi_D(f) = O\left(2^g q^{\frac{3}{2}g + \frac{3}{4}}\right). \quad (3.5.1)$$

For this we will need the following lemmas (c.f. [Fai-Rud])

**Lemma 3.5.2.** *Let  $\chi$  be a nontrivial Dirichlet character modulo  $f$ . Then for  $n < \deg(f)$ ,*

$$\left| \sum_{\deg(B)=n} \chi(B) \right| \leq \binom{\deg(f) - 1}{n} q^{n/2} \quad (3.5.2)$$

(the sum over all monic polynomials of degree  $n$ ).

*Proof.* This is a direct consequence of the Riemann Hypothesis for function fields. Comparing the series expansion of  $\mathcal{L}(u, \chi)$ , which is a polynomial of

degree at most  $\deg(f) - 1$ , with the expression in terms of the inverse zeros:

$$\sum_{0 \leq n < \deg(f)} \left( \sum_{\deg(B)=n} \chi(B) \right) u^n = \prod_{j=1}^{\deg(f)-1} (1 - \alpha_j u)$$

to get

$$\sum_{\deg(B)=n} \chi(B) = (-1)^n \sum_{\substack{S \subset \{1, \dots, \deg(f)-1\} \\ \#S=n}} \prod_{j \in S} \alpha_j,$$

and then use  $|\alpha_j| \leq \sqrt{q}$  for all  $j$ . □

**Remark 3.5.3.** Note that for  $n \geq \deg(f)$  the character sum vanishes.

Now we apply this result to quadratic characters.

**Lemma 3.5.4.** If  $f \in \mathbb{F}_q[T]$  is not a square, then

$$\left| \sum_{D \in \mathcal{H}_{2g+1, q}} \chi_D(f) \right| \ll q^{g+1/2} 2^{\deg f - 1}. \quad (3.5.3)$$

*Proof.* We use (2.2.29) to pick out the square-free monic polynomials. Thus the sum over all square-free monic polynomials is given by

$$\begin{aligned} \sum_{D \in \mathcal{H}_{2g+1, q}} \chi_D(f) &= \sum_{\deg(D)=2g+1} \sum_{A^2 | D} \mu(A) \left( \frac{D}{f} \right) \\ &= \sum_{\deg(A) \leq g} \mu(A) \left( \frac{A}{f} \right)^2 \sum_{\deg(B)=2g+1-2\deg(A)} \left( \frac{B}{f} \right). \end{aligned} \quad (3.5.4)$$

To deal with the inner sum, note that  $(\bullet/f)$  is a nontrivial character since  $f$  is not a square, so we can use Lemma 3.5.2 to obtain

$$\left| \sum_{\deg(B)=2g+1-2\deg(A)} \left( \frac{B}{f} \right) \right| \leq \binom{\deg(f) - 1}{2g + 1 - 2\deg(A)} q^{g+1/2 - \deg(A)}$$

if  $2g + 1 - 2\deg(A) < \deg(f)$ . The sum is zero otherwise. Hence we have

$$\begin{aligned}
 \left| \sum_{D \in \mathcal{H}_{2g+1,q}} \chi_D(f) \right| &\leq \sum_{\deg(A) \leq g} \left| \sum_{\deg(B)=2g+1-2\deg(A)} \left( \frac{B}{f} \right) \right| \\
 &\leq \sum_{g+1/2 - (\deg(f)/2) < \deg(A) \leq g} \binom{\deg(f) - 1}{2g + 1 - 2\deg(A)} \frac{q^{g+1/2}}{q^{\deg(A)}} \\
 &= q^{g+1/2} \sum_{g+1/2 - (\deg(f)/2) < j \leq g} \binom{\deg(f) - 1}{2g + 1 - 2j} \\
 &\leq 2^{\deg(f)-1} q^{g+1/2}.
 \end{aligned}$$

This completes the proof of Lemma 3.5.4.  $\square$

*Proof of Proposition 3.5.1.* Using Lemma 3.5.4, we have that,

$$\begin{aligned}
 \sum_{n=0}^g q^{-n/2} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f \neq \square}} \sum_{D \in \mathcal{H}_{2g+1,q}} \chi_D(f) &\leq \sum_{n=0}^g q^{-n/2} \sum_{\substack{f \text{ monic} \\ \deg(f)=n}} 2^{\deg f - 1} q^{g+1/2} \\
 &= \sum_{n=0}^g q^{-n/2} 2^{n-1} q^{g+1/2} q^n \\
 &\ll q^g \sum_{n=0}^g (q^{1/2} 2)^n \\
 &\ll q^g (2q^{1/2})^{g+1} \\
 &\ll q^{\frac{3}{2}g + \frac{3}{4}} 2^g. \tag{3.5.5}
 \end{aligned}$$

$\square$

## 3.6 Proof of the Main Theorem

*Proof of Theorem 3.1.1.* Now we are in a position to prove Theorem 3.1.1. For this we make use of Proposition 3.4.1 and Proposition 3.5.1, which give us

$$\begin{aligned}
 &\sum_{D \in \mathcal{H}_{2g+1,q}} \sum_{n=0}^g \sum_{\substack{f_1 \text{ monic} \\ \deg(f_1)=n}} \chi_D(f_1) q^{-n/2} \\
 &= \frac{P(1)}{\zeta_A(2)} |D| \left\{ ([g/2] + 1) + \sum_P \frac{\deg(P)}{|P|(|P| + 1) - 1} \right\} + O\left(2^g q^{\frac{3}{2}g + \frac{3}{4}}\right). \tag{3.6.1}
 \end{aligned}$$

For the dual sum in (3.3.2) we get, similarly, that

$$\begin{aligned} & \sum_{D \in \mathcal{H}_{2g+1, q}} \sum_{m=0}^{g-1} \sum_{\substack{f_2 \text{ monic} \\ \deg(f_2)=m}} \chi_D(f_2) q^{-m/2} \\ &= \frac{P(1)}{\zeta_A(2)} |D| \left\{ \left( \left[ \frac{g-1}{2} \right] + 1 \right) + \sum_P \frac{\deg(P)}{|P|(|P|+1)-1} \right\} + O\left(2^g q^{\frac{3}{2}g + \frac{3}{4}}\right). \end{aligned} \quad (3.6.2)$$

So, adding (3.6.1) with (3.6.2), we see that,

$$\begin{aligned} & \sum_{D \in \mathcal{H}_{2g+1, q}} \mathcal{L}(q^{-1/2}, \chi_D) = \\ & \frac{P(1)}{2\zeta_A(2)} |D| \left\{ \log_q |D| + 1 + 4 \sum_P \frac{\deg(P)}{|P|(|P|+1)-1} \right\} + O\left(2^g q^{\frac{3}{2}g + \frac{3}{4}}\right) \end{aligned} \quad (3.6.3)$$

and using the fact that  $|D| = q^{2g+1}$  and that

$$\frac{4}{\log q} \frac{P'}{P}(1) = 4 \sum_{\substack{P \text{ monic} \\ \text{irreducible}}} \frac{\deg(P)}{|P|(|P|+1)-1},$$

we have precisely the statement of Theorem 3.1.1.  $\square$

Note that if we let  $q \rightarrow \infty$  the error term in the Theorem 3.1.1 becomes  $O(|D|^{3/4+\varepsilon})$ , which now appears precisely in the same form as the error term in Jutila's result for the number field case.



## Chapter 4

# A Mean Value Theorem over Monic Irreducible Polynomials in $\mathbb{F}_q[T]$

In this chapter, we will mimic the calculations of Chapter 3 to establish the function field analogue of the following result due to Jutila [Jutila]

**Theorem 4.0.1** (Jutila). *We have that,*

$$\begin{aligned} & \sum_{\substack{p \leq X \\ p \equiv 3 \pmod{4} \\ p \text{ prime}}} (\log p) L\left(\frac{1}{2}, \chi_p\right) \\ &= \frac{1}{4} X \left\{ \log(X/\pi) + \frac{\Gamma'}{\Gamma}\left(\frac{3}{4}\right) + 4\gamma - 1 \right\} + O(X(\log X)^{-A}), \end{aligned} \quad (4.0.1)$$

where the implied constant is not effectively calculable. The following estimate is effective:

$$\sum_{\substack{p \leq X \\ p \equiv 3 \pmod{4} \\ p \text{ prime}}} (\log p) L\left(\frac{1}{2}, \chi_p\right) = \frac{1}{4} X \log X + O(X(\log X)^\varepsilon), \quad (4.0.2)$$

where  $\gamma$  is the Euler–Mascheroni constant.

The function field analogue of this theorem which we established in this chapter is

**Theorem 4.0.2.** *Let  $\mathbb{F}_q$  be a finite field where  $q \equiv 1 \pmod{4}$  is fixed. And let  $P \in \mathbb{F}_q[T]$  be a monic irreducible polynomial such that  $\deg(P) = 2M + 1$ . Then we have,*

$$\begin{aligned} & \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1}} (\log |P|) L\left(\frac{1}{2}, \chi_P\right) \\ &= \frac{|P|}{2} \left\{ \log |P| + \log q - \frac{\log |P| + \log q}{|P|} \right\} + O(|P|^{3/4+\varepsilon}). \end{aligned} \quad (4.0.3)$$

We now present a direct corollary from Theorem 4.0.2

**Corollary 4.0.3.** *Under the same hypothesis of the above theorem, we have that,*

$$L\left(\frac{1}{2}, \chi_P\right) \neq 0 \quad (4.0.4)$$

for infinitely many monic irreducible polynomials  $P$  of degree odd.

## 4.1 Setting Up the Problem

Let  $P$  be a monic irreducible polynomial of degree odd, i.e.,  $\deg(P) = 2M + 1$  and let  $C_P$  be a smooth, projective and geometrically connected curve of genus  $g \geq 1$  defined over the finite field  $\mathbb{F}_q$ . The zeta-function associated with the model  $C_P : y^2 = P(x)$  is a rational function of the form

$$Z_{C_P}(u) = \frac{\mathcal{L}(u, \chi_P)}{(1-u)(1-qu)}, \quad |u| < 1/q \quad (4.1.1)$$

where  $\mathcal{L}(u, \chi_P)$  is a polynomial of degree  $2g = 2M$  and  $\mathcal{L}(u, \chi_P) \in \mathbb{Z}[u]$  with  $\mathcal{L}(0, \chi_P) = 1$  and satisfies the functional equation

$$\mathcal{L}(u, \chi_P) = (qu^2)^g \mathcal{L}\left(\frac{1}{qu}, \chi_P\right). \quad (4.1.2)$$

Using the same ideas, as we did in Chapter 3, we can write  $\mathcal{L}(u, \chi_P)$  as follows

$$\begin{aligned} \mathcal{L}(q^{-\frac{1}{2}}, \chi_P) &= L(\tfrac{1}{2}, \chi_P) \\ &= \sum_{n=0}^M \sum_{\substack{f_1 \text{ monic} \\ \deg(f_1)=n}} \chi_P(f_1) q^{-n/2} + \sum_{m=0}^{M-1} \sum_{\substack{f_2 \text{ monic} \\ \deg(f_2)=m}} \chi_P(f_2) q^{-m/2}. \end{aligned} \quad (4.1.3)$$

Our main goal in this chapter is to establish an asymptotic formula for

$$\sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1}} (\log |P|) L(\tfrac{1}{2}, \chi_P) \quad (4.1.4)$$

in the limit  $M \rightarrow \infty$ . And we will do this by using (4.1.3).

Now we present the details of the calculation for the average of the first term in the right-hand side of (4.1.3) and since we have that the terms are similar we need only worry about computing one of them to obtain the final result. Therefore, we want an asymptotic formula for

$$\begin{aligned} & \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1}} (\log |P|) \sum_{n=0}^M \sum_{\substack{f \text{ monic} \\ \deg(f)=n}} \chi_P(f) q^{-n/2} \\ &= (2M+1)(\log q) \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1}} \sum_{n=0}^M \sum_{\substack{f \text{ monic} \\ \deg(f)=n}} \chi_P(f) q^{-n/2} \\ &= (2M+1)(\log q) \\ & \times \left( \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1}} \sum_{n=0}^M \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f=\square}} \frac{\chi_P(f)}{q^{n/2}} + \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1}} \sum_{n=0}^M \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f \neq \square}} \frac{\chi_P(f)}{q^{n/2}} \right). \end{aligned} \quad (4.1.5)$$

The first term (the square contributions) will provide us with the main term and the second term in (4.1.5) (the nonsquare contributions) will be bounded using the Riemann Hypothesis for curves [Weil-CAD].

## 4.2 The Main Term Calculation

In this section we will establish the main term of Theorem 4.0.2. For this we will prove two auxiliary propositions about the square contribution for the average and the main tool used will be the Polynomial Prime Number Theorem. Note that our result also includes the main lower order terms.

**Proposition 4.2.1.** *We have that,*

$$\begin{aligned} & \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1}} \sum_{n=0}^M \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f=\square}} \chi_P(f) q^{-n/2} \\ &= \left( \left[ \frac{M}{2} \right] + 1 \right) \left( \frac{|P|}{\log_q |P|} - \frac{1}{\log_q |P|} \right) + O \left( \sqrt{|P|} \left( \left[ \frac{M}{2} \right] + 1 \right) \right). \end{aligned} \quad (4.2.1)$$

*Proof.* We start writing

$$\begin{aligned} & \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1}} \sum_{n=0}^M \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f=m^2}} \chi_P(f) q^{-n/2} = \sum_{n=0}^M q^{-n/2} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f=m^2}} \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1}} \chi_P(f) \\ &= \sum_{\substack{n=0 \\ 2|n}}^M q^{-n/2} \sum_{\substack{m \text{ monic} \\ \deg(m)=n/2}} \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1}} \chi_P(m^2) = \sum_{\substack{n=0 \\ 2|n}}^M q^{-n/2} \sum_{\substack{m \text{ monic} \\ \deg(m)=n/2}} \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1 \\ (P,m)=1}} 1 \\ &= \sum_{\substack{n=0 \\ 2|n}}^M q^{-n/2} \sum_{\substack{m \text{ monic} \\ \deg(m)=n/2}} \left( \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1}} 1 - \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1 \\ P|m}} 1 \right) \\ &= \sum_{\substack{n=0 \\ 2|n}}^M q^{-n/2} \sum_{\substack{m \text{ monic} \\ \deg(m)=n/2}} \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1}} 1 - \sum_{\substack{n=0 \\ 2|n}}^M q^{-n/2} \sum_{\substack{m \text{ monic} \\ \deg(m)=n/2}} \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1 \\ P|m}} 1, \end{aligned} \quad (4.2.2)$$

and making use of the Polynomial Prime Number Theorem 2.1.5 we obtain

that (4.2.2) is,

$$\begin{aligned}
 &= \sum_{\substack{n=0 \\ 2|n}}^M q^{-n/2} \sum_{\substack{m \text{ monic} \\ \deg(m)=n/2}} \left( \frac{q^{2M+1}}{2M+1} + O\left(q^{\frac{2M+1}{2}}\right) \right) \\
 &\quad - \sum_{\substack{n=0 \\ 2|n}}^M q^{-n/2} \sum_{\substack{P \text{ monic} \\ \deg(P)=2M+1}} \sum_{\substack{m \text{ monic} \\ \deg(m)=n/2 \\ P|m}} 1 \\
 &= \sum_{\substack{n=0 \\ 2|n}}^M q^{-n/2} \sum_{\substack{m \text{ monic} \\ \deg(m)=n/2}} \frac{q^{2M+1}}{2M+1} - \sum_{\substack{n=0 \\ 2|n}}^M q^{-n/2} \sum_{\substack{P \text{ monic} \\ \deg(P)=2M+1}} \sum_{\substack{a \text{ monic} \\ \deg(a)=\frac{n}{2}-2M-1}} 1 \\
 &\quad + O\left( \sum_{\substack{n=0 \\ 2|n}}^M q^{-n/2} \sum_{\substack{m \text{ monic} \\ \deg(m)=n/2}} q^{\frac{2M+1}{2}} \right) \\
 &= \frac{q^{2M+1}}{2M+1} \sum_{\substack{n=0 \\ 2|n}}^M 1 - \sum_{\substack{n=0 \\ 2|n}}^M q^{-n/2} q^{n/2} q^{-2M-1} \sum_{\substack{P \text{ monic} \\ \deg(P)=2M+1}} 1 + O\left( q^{\frac{2M+1}{2}} \sum_{\substack{n=0 \\ 2|n}}^M 1 \right) \\
 &= \frac{q^{2M+1}}{2M+1} \sum_{n=0}^{\lfloor \frac{M}{2} \rfloor} 1 - q^{-2M-1} \sum_{n=0}^{\lfloor \frac{M}{2} \rfloor} \left( \frac{q^{2M+1}}{2M+1} + O\left(q^{\frac{2M+1}{2}}\right) \right) + O\left( q^{\frac{2M+1}{2}} \sum_{n=0}^{\lfloor \frac{M}{2} \rfloor} 1 \right) \\
 &= \frac{q^{2M+1}}{2M+1} \left( \left[ \frac{M}{2} \right] + 1 \right) - \frac{q^{-2M-1} q^{2M+1}}{2M+1} \left( \left[ \frac{M}{2} \right] + 1 \right) \\
 &\quad + O\left( q^{\frac{2M+1}{2}} q^{-2M-1} \left( \left[ \frac{M}{2} \right] + 1 \right) \right) + O\left( q^{\frac{2M+1}{2}} \sum_{n=0}^{\lfloor \frac{M}{2} \rfloor} 1 \right) \\
 &= \frac{q^{2M+1}}{2M+1} \left( \left[ \frac{M}{2} \right] + 1 \right) - \frac{1}{2M+1} \left( \left[ \frac{M}{2} \right] + 1 \right) + O\left( q^{\frac{2M+1}{2}} \left( \left[ \frac{M}{2} \right] + 1 \right) \right),
 \end{aligned}$$

which proves the proposition.  $\square$

Similarly, we are able to prove the following proposition about the dual sum

**Proposition 4.2.2.** *Under the same conditions of the above proposition we have,*

$$\begin{aligned} & \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1}} \sum_{n=0}^{M-1} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f \neq \square}} \chi_P(f) q^{-n/2} \\ &= \left( \left[ \frac{M-1}{2} \right] + 1 \right) \left( \frac{|P|}{\log_q |P|} - \frac{1}{\log_q |P|} \right) + O \left( \sqrt{|P|} \left( \left[ \frac{M-1}{2} \right] + 1 \right) \right). \end{aligned} \quad (4.2.3)$$

Putting the propositions 4.2.1 and 4.2.2 together and multiplying the result by  $\log |P|$  we obtain the main term given in the statement of Theorem 4.0.2

$$\begin{aligned} \text{Main-Term} &= \frac{\log q}{2} |P| \left\{ \log_q |P| + 1 - \frac{\log_q |P| + 1}{|P|} \right\} \\ &+ O \left( \log |P| \sqrt{|P|} \left( \left[ \frac{M}{2} \right] + 1 \right) \right). \end{aligned} \quad (4.2.4)$$

### 4.3 Bounding Character Sums Over Primes

In this section we establish the following estimate

**Proposition 4.3.1.** *We have that,*

$$\sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1}} \sum_{n=0}^M \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f \neq \square}} \chi_P(f) q^{-n/2} = O \left( q^{\frac{3M}{2}} \right). \quad (4.3.1)$$

To establish this estimate we will relate  $\sum_P \left( \frac{f}{P} \right)$  for a fixed nonsquare  $f$  to a quantity bounded by the Riemann Hypothesis for function fields. The result that we will use for the proof of this proposition is

**Theorem 4.3.2** (c.f. [Rud-TPFHE]). *Assume that  $B$  is monic, of positive degree and not a perfect square. Then we have the following bound for the*

character sum over prime polynomials:

$$\left| \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=n}} \left( \frac{B}{P} \right) \right| \ll \frac{\deg(B)}{n} q^{n/2}. \quad (4.3.2)$$

*Proof.* Remembering that we can write,

$$\mathcal{L}(u, \chi_P) = \det(I - u\sqrt{q}\Theta_{C_P}), \quad (4.3.3)$$

where  $\Theta_{C_P} \in USp(2M)$ , then we have the following explicit formula

$$-\text{tr}\Theta_{C_P}^n = \frac{1}{q^{n/2}} \sum_{\substack{f \text{ monic} \\ \deg(f)=n}} \Lambda(f)\chi_P(f). \quad (4.3.4)$$

Now we can write  $B = DC^2$  where  $D$  is square-free polynomial, of positive degree and, together with (4.3.4) and the unitarity of  $\Theta_{C_P}$  (which is the Riemann Hypothesis for curves) assures us the expected result.  $\square$

*Proof of Proposition 4.3.1.*

$$\begin{aligned} & \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1}} \sum_{n=0}^M \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f \neq \square}} \chi_P(f)q^{-n/2} = \sum_{n=0}^M q^{-n/2} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f \neq \square}} \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1}} \chi_P(f) \\ & = \sum_{n=0}^M q^{-n/2} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f \neq \square}} \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1}} \left( \frac{P}{f} \right). \end{aligned} \quad (4.3.5)$$

Using the quadratic reciprocity law for  $\mathbb{F}_q[T]$  (see Chapter 2) we have that

$$\left( \frac{P}{f} \right) = \left( \frac{f}{P} \right) (-1)^{\frac{(q-1)}{2}(2M+1)n}, \quad (4.3.6)$$

and as  $(-1)^{\frac{(q-1)}{2}(2M+1)n}$  has the same sign for all  $P$  such that  $\deg(P) = 2M+1$  and  $f$  monic, then we have

$$\left| \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1}} \left( \frac{P}{f} \right) \right| = \left| \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1}} \left( \frac{f}{P} \right) \right|. \quad (4.3.7)$$

So we can write,

$$\sum_{n=0}^M q^{-n/2} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f \neq \square}} \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1}} \left( \frac{P}{f} \right) \leq \sum_{n=0}^M q^{-n/2} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f \neq \square}} \left| \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1}} \left( \frac{f}{P} \right) \right|. \quad (4.3.8)$$

Using now the Theorem 4.3.2 we have that the quantity in (4.3.8) is,

$$\begin{aligned} \ll \sum_{n=0}^M q^{-n/2} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f \neq \square}} \frac{n}{2M+1} q^{\frac{2M+1}{2}} &\ll \sum_{n=0}^M q^{-n/2} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f \neq \square}} n q^{\frac{2M+1}{2}} \\ &\ll \sum_{n=0}^M n q^{n/2} q^{\frac{2M+1}{2}} \\ &= q^{\frac{2M+1}{2}} \sum_{n=0}^M n q^{n/2}, \end{aligned} \quad (4.3.9)$$

and making use of the arithmetic-geometric progression formula given by the equation 0.113 in [Gra-Ryz]

$$\sum_{k=0}^{n-1} (a + kr)q^k = \frac{a - [a + (n-1)r]q^n}{1-q} + \frac{rq(1-q^{n-1})}{(1-q)^2},$$

we have that

$$\begin{aligned} \sum_{n=0}^M q^{-n/2} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f \neq \square}} \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1}} \left( \frac{P}{f} \right) &\ll q^{\frac{2M+1}{2}} (\sqrt{q} + (\sqrt{q})^{M+1}) \\ &\ll (\sqrt{q})^{3M+2} \\ &\ll \frac{3M}{q^2}, \end{aligned} \quad (4.3.10)$$

which concludes the proof of the proposition, as desired.  $\square$

## 4.4 Proof of the Main Theorem

Now we are in a position to prove the main theorem of this chapter.

*Proof of Theorem 4.0.2.*

$$\sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1}} (\log |P|) L\left(\frac{1}{2}, \chi_P\right) = (2M+1)(\log q) \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1}} L\left(\frac{1}{2}, \chi_P\right). \quad (4.4.1)$$



Using (4.2.4) and Proposition 4.3.1 we have that

$$\begin{aligned}
 & (2M+1)(\log q) \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1}} L\left(\frac{1}{2}, \chi_P\right) \\
 &= (2M+1)(\log q) \left( \frac{|P|}{\log_q |P|} (M+1) - \frac{1}{\log_q |P|} (M+1) \right) \\
 &\quad + (2M+1)(\log q) O\left(\sqrt{|P|} \left(\left[\frac{M}{2}\right] + 1\right)\right) + O\left((2M+1)(\log q) q^{\frac{3M}{2}}\right) \\
 &= (\log q)(|P|(M+1) - (M+1)) + O(|P|^{3/4} \log_q |P|). \tag{4.4.2}
 \end{aligned}$$

Making the following substitution

$$(M+1) = \frac{\log_q |P|}{2} + \frac{1}{2}, \tag{4.4.3}$$

in the above equation, we have that (4.4.2) becomes

$$\frac{\log q}{2} |P| \left\{ \log_q |P| + 1 - \frac{\log_q |P| + 1}{|P|} \right\} + O(|P|^{3/4+\varepsilon}), \tag{4.4.4}$$

which is precisely the statement of our main theorem in this chapter, since

$$(\log q)(\log_q |P|) = \log |P|. \tag{4.4.5}$$

□

# Chapter 5

## Asymptotic Averages for the Class Numbers over Function Fields

In this chapter, we investigate the class number problem in the context of function fields making use of the techniques developed in Chapters 3 and 4.

### 5.1 The Class Number in the Number Field Setting

The class number problem begins with Gauss, which in his famous *Disquisitiones Arithmeticae* [Gauss-DA] presented two conjectures concerning the average values of these mysterious numbers that are associated with binary quadratic forms  $ax^2 + 2bxy + cy^2$  where  $a, b, c \in \mathbb{Z}$ . For completeness, clarity and to put our problem in the right context we will restate the Gauss's conjectures.

Let  $D = 4(b^2 - ac)$  be the discriminant of the quadratic form  $ax^2 + 2bxy + cy$ . Gauss considered only even discriminants (due to the restriction of the coefficient of  $xy$  be even) and then he defined an equivalence relation between

the quadratic forms as follows: We say that two quadratic forms are equivalent if it is possible to transform the first form into the second through an invertible integral linear change of variables. This is an equivalence relation on the set of quadratic forms and the equivalence classes will be called classes of quadratic forms. Gauss showed that the number of equivalence classes of quadratic forms with discriminant  $D$  is finite. Let  $h_D$  denote this number, we also call  $h_D$  the **class number**. We now present Gauss's conjectures concerning mean values of  $h_D$  quoted from [Hoff-Ros].

**Conjecture 5.1.1** (Gauss). *Let  $h_D$  be the class number as above. Then we have,*

1. *Let  $D = -4k$  run over all negative discriminants with  $k \leq X$ . Then*

$$\sum_{1 \leq k \leq X} h_D \sim \frac{4\pi}{21\zeta(3)} X^{\frac{3}{2}}. \quad (5.1.1)$$

2. *Let  $D = 4k$  run over all positive discriminants such that  $k \leq X$ . Then*

$$\sum_{1 \leq k \leq X} h_D R_D \sim \frac{4\pi^2}{21\zeta(3)} X^{\frac{3}{2}}, \quad (5.1.2)$$

where  $R_D = \log(\varepsilon_D)$  with  $\varepsilon_D$  the regulator of the real quadratic number field  $\mathbb{Q}(\sqrt{D})$ .

The first conjecture (5.1.1) was proved by Lipschitz [Lipschitz] and the second one (5.1.2) by Siegel [Siegel]. We can reformulate the above conjectures in terms of orders  $\mathcal{O}$  in the quadratic number fields as follows: Let  $D \equiv 0, 1 \pmod{4}$  such that  $D$  is not a perfect square. Then  $D = D_0 m^2$  where  $D_0, m \in \mathbb{Z}$  and we have either  $D_0$  is square-free and  $D_0 \equiv 1 \pmod{4}$  or  $D_0 = 4m_0$  with  $m_0 \equiv 2, 3 \pmod{4}$  and square-free. Now, if  $m = 1$ , we call  $D$  a fundamental discriminant, as was done in Section 1.2.3. Consider the quadratic number field  $\mathbb{Q}(\sqrt{D})$ , then there is a unique order  $\mathcal{O}_D \subset \mathcal{O}_{D_0}$  (called maximal order) where the maximal order has discriminant  $D_0$  and such that  $[\mathcal{O}_{D_0} : \mathcal{O}_D] = m$ . In this case, let  $h_D$  denote the strict class number of  $\mathcal{O}_D$  and

for  $D > 0$ , let  $\varepsilon_D$  denote the smallest unit in  $\mathcal{O}_D$  such that  $\varepsilon_D > 1$  and  $\mathbb{N}_{\mathbb{Q}(\sqrt{D})/\mathbb{Q}}(\varepsilon_D) = 1$  with  $R_D = \log(\varepsilon_D)$ . In other words, the class number  $h_D$  is related and can be seen in terms of the size of invertible fractional ideals of  $\mathcal{O}$  modulo principal ideals, i.e, the Picard group of  $\mathcal{O}$ , which we denote by  $\text{Pic}(\mathcal{O})$ .

Siegel also showed the following result on the class numbers, where the average is taken over all discriminants

**Theorem 5.1.2** (Siegel). *We have,*

1.

$$\sum_{1 < -D \leq X} h_D = \frac{\pi}{18\zeta(3)} X^{3/2} + O(X \log X), \quad (5.1.3)$$

2.

$$\sum_{1 < D \leq X} h_D R_D = \frac{\pi^2}{18\zeta(3)} X^{3/2} + O(X \log X). \quad (5.1.4)$$

Let  $\psi_D(n) = \left(\frac{D}{n}\right)$ , be the Kronecker's symbol and  $L(s, \psi_D)$  the Dirichlet series associated with  $\psi_D(n)$ , then can be deduced from Siegel's paper that

$$\sum_{1 < \pm D \leq X} L(1, \psi_D) = \frac{1}{2} \frac{\zeta(2)}{\zeta(3)} X + O(X^{1/2} \log X). \quad (5.1.5)$$

Now, using the following result due to Dirichlet

$$L(1, \psi_D) = \begin{cases} \frac{2\pi h_D}{w_D \sqrt{D}} & \text{if } D < 0, \\ \frac{h_D R_D}{\sqrt{D}} & \text{if } D > 0, \end{cases}$$

together with the average of the  $L$ -function at  $s = 1$  and partial summation we have that Theorem 5.1.2 can be deduced. Note that  $w_D = 2$  except when  $D = -4$  or  $-6$ , when  $w_D = 4$  and  $6$  respectively.

## 5.2 The Function Field Case

We now move to the function field analogue of the class number problem. As usual, we will make the natural change from  $\mathbb{Z}$  and  $\mathbb{Q}$  to  $A = \mathbb{F}_q[T]$  and

$k = \mathbb{F}_q(T)$  respectively, where  $A$  is the polynomial ring over  $\mathbb{F}_q$  and  $k$  is the rational function field over  $\mathbb{F}_q$ . From now on we assume that the cardinality of  $\mathbb{F}_q$  is odd and fixed with  $q \equiv 1 \pmod{4}$ .

Now we mimic the discussion made in the previous section for the function field case. Let  $K = k(\sqrt{m})$ , where  $m \in A$  is a non-square polynomial. So we can write,  $m = Dm_1^2$ , where  $D$  is a square-free polynomial. Let  $\mathcal{O}_m = A + A\sqrt{m} \subset K$ , then  $\mathcal{O}_m$  is an  $A$ -order in  $K$ . The Picard group of  $\mathcal{O}_m$ ,  $\text{Pic}(\mathcal{O}_m)$ , is in this setting, in the same way as it is in the number field case, the group of invertible fractional ideals of  $\mathcal{O}_m$  modulo principal fractional ideals. And the class number  $h_m$  is now defined to be  $h_m = |\text{Pic}(\mathcal{O}_m)|$ .

Let  $K/k$  be a quadratic extension and call  $\mathcal{O}_K$  the integral closure of  $A$  in  $K$ . Consider that  $D \in A$  is a square-free polynomial and put  $\mathcal{O}_K$  as  $\mathcal{O}_D = A[\sqrt{D}]$ . So in this case,  $\mathcal{O}_D$  is a Dedekind domain and  $\text{Pic}(\mathcal{O}_D) = \text{Cl}(\mathcal{O}_D)$ , where  $\text{Cl}(\mathcal{O}_D)$  is the class group of  $\mathcal{O}_D$ . We will denote the class number for this case as  $h_D$ , and by [Rosen, Proposition 14.2] we have that  $h_D$  is finite.

Hoffstein and Rosen [Hoff-Ros] succeeded in compute the average value of the class number  $h_D$  when the average is taken over all monic polynomials  $D$  of a fixed degree, they showed that

**Theorem 5.2.1** (Hoffstein and Rosen). *If  $M$  is odd and positive. Then,*

$$\frac{1}{q^M} \sum_{\substack{D \text{ monic} \\ \deg(D)=M}} h_D = \frac{\zeta_A(2)}{\zeta_A(3)} q^{\frac{M-1}{2}} - q^{-1}. \quad (5.2.1)$$

The theorem above can be seen as the function field analogue of the Siegel's theorem. They also showed in the same paper the following result

**Theorem 5.2.2.** *Let  $M$  be even and positive and the following sum is over all non-square monic polynomials of degree  $M$ . Then we have,*

$$\begin{aligned} & \frac{1}{q^M} \sum h_D R_D \\ &= (q-1)^{-1} \left( \frac{\zeta_A(2)}{\zeta_A(3)} q^{M/2} - \left( 2 + \left( 1 - \frac{1}{q} \right) (M-1) \right) \right). \quad (5.2.2) \end{aligned}$$

A problem which is more difficult and we consider in this chapter is to calculate the average value of the class numbers over fundamental discriminants, i.e.,  $D$  monic and square-free. We should note that the calculations presented in this chapter follow the same philosophy of the calculations presented in Chapter 3, where we fix the number of elements of the finite field and compute the limit as  $\deg(D) \rightarrow \infty$  to obtain our asymptotic formulas.

### 5.3 Statement of Results

As we said earlier, the class numbers  $h_D$  is equal to the  $|\text{Pic}(\mathcal{O}_D)|$ , where  $\text{Pic}(\mathcal{O}_D)$  is the Picard group of  $\mathcal{O}_D$ . But we also should note that if  $D \in \mathcal{H}_{2g+1,q}$ , then the equation  $y^2 = D(T)$  defines a hyperelliptic curve  $C_D$  over  $\mathbb{F}_q$  of genus  $g$  and the number  $h_D$  is closely related to the set of the  $\mathbb{F}_q$ -rational points on its Jacobian,  $\text{Jac}(C_D)$ , and so our result also has a geometric appeal. With this in mind, the main theorems of this chapter which we will establish in the subsequent sections are

**Theorem 5.3.1.** *Let  $D \in \mathcal{H}_{2g+1,q}$ . Then we have,*

$$\frac{1}{\#\mathcal{H}_{2g+1,q}} \sum_{D \in \mathcal{H}_{2g+1,q}} h_D \sim \frac{\sqrt{|D|}}{\sqrt{q}} \zeta_A(2) P(2), \quad (5.3.1)$$

as  $\deg(D) \rightarrow \infty$ , i.e.,  $g \rightarrow \infty$ . Where

$$P(s) = \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left( 1 - \frac{1}{(|P|+1)|P|^s} \right). \quad (5.3.2)$$

**Theorem 5.3.2.** *Let  $P \in A$  be a monic irreducible polynomial of degree  $2M+1$  and  $h_P$  be the associated class number. Then we have,*

$$\frac{1}{\Pi_q(2M+1)} \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1}} h_P \sim \frac{1}{\sqrt{q}} \zeta_A(2) \sqrt{|P|}, \quad (5.3.3)$$

as  $M \rightarrow \infty$ . Where  $\Pi_q(M)$  is the number of monic irreducible polynomials of degree  $M$ .

The Theorem 5.3.1 also appears in [Andrade].

## 5.4 Preparation for the Proof of Main Theorems

In this section, we establish the following propositions

**Proposition 5.4.1.** *Let  $\mathbb{F}_q$  be a fixed finite field with  $q \equiv 1 \pmod{4}$ . Then*

$$\begin{aligned} & \sum_{D \in \mathcal{H}_{2g+1,q}} L(1, \chi_D) \\ &= |D| \left\{ P(2) - P(1) \left( \frac{1}{q^{\lfloor g/2 \rfloor + 1}} + \frac{1}{\zeta_A(2)^2 q^{g \lfloor (g-1)/2 \rfloor}} \right) \right\} + O((2q)^g), \end{aligned} \quad (5.4.1)$$

where  $|D| = q^{2g+1}$  and

$$P(s) = \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left( 1 - \frac{1}{(|P| + 1)|P|^s} \right). \quad (5.4.2)$$

As a corollary of the Proposition 5.4.1 we have that,

**Corollary 5.4.2.**

$$\frac{1}{\#\mathcal{H}_{2g+1,q}} \sum_{D \in \mathcal{H}_{2g+1,q}} L(1, \chi_D) \sim \zeta_A(2)P(2) \quad (5.4.3)$$

as  $\deg(D) \rightarrow \infty$ , i.e.,  $g \rightarrow \infty$ .

*Proof.* Using the Theorem 5.4.1 together with the formula (2.2.27) for the number of elements in the hyperelliptic ensemble  $\mathcal{H}_{2g+1,q}$  and computing the limit as  $g \rightarrow \infty$ , we can conclude the asymptotic formula above.  $\square$

We also establish the following result

**Proposition 5.4.3.** *We have that,*

$$\begin{aligned} & \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1}} L(1, \chi_P) \\ &= \frac{1}{(2M+1)(q-1)} \left( q^{2M+1} (q^{-\lfloor M/2 \rfloor} (q^{1+\lfloor M/2 \rfloor} - 1)) - q^{-\lfloor M/2 \rfloor} (q^{1+\lfloor M/2 \rfloor} - 1) \right. \\ & \quad \left. + q^{M+1} \left( q^{1+\lfloor \frac{M-1}{2} \rfloor} - 1 \right) - q^{-M} \left( q^{1+\lfloor \frac{M-1}{2} \rfloor} - 1 \right) \right) + O\left(\frac{q^M M^2}{2M+1}\right). \end{aligned} \quad (5.4.4)$$

As a corollary of the Proposition 5.4.3 we have that,

**Corollary 5.4.4.**

$$\frac{1}{\prod_q(2M+1)} \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1}} L(1, \chi_P) \sim \frac{|P|}{\log_q |P|} \zeta_A(2) \quad (5.4.5)$$

as  $\deg(P) = M \rightarrow \infty$ .

*Proof.* Using the Proposition 5.4.3 together with the Polynomial Prime Number Theorem given in Section 2.1.2 and computing the limit as  $M \rightarrow \infty$ , we can conclude the asymptotic formula above.  $\square$

The starting point for the proof of the main results is the following representation for  $L(s, \chi_D)$

$$L(s, \chi_D) = \sum_{\substack{f_1 \text{ monic} \\ \deg(f_1) \leq g}} \frac{\chi_D(f_1)}{|f_1|^s} + (q^{1-2s})^g \sum_{\substack{f_2 \text{ monic} \\ \deg(f_2) \leq g-1}} \frac{\chi_D(f_2)}{|f_2|^{1-s}}. \quad (5.4.6)$$

### 5.4.1 Proof of Proposition 5.4.1

#### Preliminary Lemmas

We will require some auxiliary lemmas. We begin with a bound for nontrivial character sums using the Lemma 3.5.2, which is a consequence of the Riemann Hypothesis for function fields.

**Lemma 5.4.5.** *We have that,*

1.

$$\sum_{D \in \mathcal{H}_{2g+1, q}} \sum_{n=0}^g q^{-n} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f \neq \square}} \chi_D(f) \ll (2q)^g. \quad (5.4.7)$$

2.

$$q^{-g} \sum_{D \in \mathcal{H}_{2g+1, q}} \sum_{m=0}^{g-1} \sum_{\substack{f \text{ monic} \\ \deg(f)=m \\ f \neq \square}} \chi_D(f) \ll (2q)^g. \quad (5.4.8)$$



*Proof.* We will establish the Part (1) of this Lemma. For Part (2) we have that the proof is analogous. We start with

$$\begin{aligned}
 & \sum_{D \in \mathcal{H}_{2g+1,q}} \sum_{n=0}^g q^{-n} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f \neq \square}} \chi_D(f) \\
 &= \sum_{n=0}^g q^{-n} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f \neq \square}} \sum_{\substack{D \text{ monic} \\ \deg(D)=2g+1}} \sum_{\substack{A \text{ monic} \\ A^2 | D}} \mu(A) \left( \frac{D}{f} \right) \\
 &= \sum_{n=0}^g q^{-n} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f \neq \square}} \sum_{\substack{A \text{ monic} \\ \deg(A) \leq g}} \mu(A) \left( \frac{A}{f} \right)^2 \sum_{\substack{B \text{ monic} \\ \deg(B)=2g+1-2\deg(A)}} \left( \frac{B}{f} \right).
 \end{aligned}$$

Now we note that  $(B/f)$  is a nontrivial character since  $f \neq \square$ . So we can invoke the Lemma 3.5.2 to get the following bound

$$\left| \sum_{\substack{B \text{ monic} \\ \deg(B)=2g+1-2\deg(A)}} \left( \frac{B}{f} \right) \right| \leq \binom{\deg(f) - 1}{2g + 1 - 2\deg(A)} q^{g + \frac{1}{2} - \deg(A)} \quad (5.4.9)$$

if  $2g + 1 - 2\deg(A) < \deg(f)$ , and the sum is zero otherwise. So we have,

$$\begin{aligned}
 & \sum_{D \in \mathcal{H}_{2g+1,q}} \sum_{n=0}^g q^{-n} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f \neq \square}} \chi_D(f) \\
 & \leq \sum_{n=0}^g q^{-n} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f \neq \square}} \sum_{\substack{A \text{ monic} \\ \deg(A) \leq g}} \mu(A) \left| \sum_{\substack{B \text{ monic} \\ \deg(B)=2g+1-2\deg(A)}} \left( \frac{B}{f} \right) \right| \\
 & \leq \sum_{n=0}^g q^{-n} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f \neq \square}} \sum_{j=g+\frac{1}{2}-\frac{\deg(f)}{2}}^g \sum_{\substack{A \text{ monic} \\ \deg(A)=j}} \binom{\deg(f) - 1}{2g + 1 - 2\deg(A)} q^{g + \frac{1}{2} - \deg(A)} \\
 & \ll q^g \sum_{n=0}^g q^{-n} \sum_{\substack{f \text{ monic} \\ \deg(f)=n}} \sum_{j=g+\frac{1}{2}-\frac{\deg(f)}{2}}^g \binom{\deg(f) - 1}{2g + 1 - 2j} \\
 & \ll q^g \sum_{n=0}^g q^{-n} \sum_{\substack{f \text{ monic} \\ \deg(f)=n}} 2^{\deg(f)-1} = q^g \sum_{n=0}^g 2^n \ll q^g 2^g.
 \end{aligned}$$

□

We will now state and prove our next two lemmas.

**Lemma 5.4.6.** For  $|D| = q^{2g+1}$  we have that,

$$|D| \sum_{\substack{d \text{ monic} \\ \deg(d) > [g/2]}} \frac{\mu(d)}{|d|^2} \prod_{\substack{P \text{ monic} \\ \text{irreducible} \\ P|d}} \frac{1}{|P|+1} \ll q^g. \quad (5.4.10)$$

*Proof.*

$$\begin{aligned} |D| \sum_{\substack{d \text{ monic} \\ \deg(d) > [g/2]}} \frac{\mu(d)}{|d|^2} \prod_{P|d} \frac{1}{|P|+1} &\leq |D| \sum_{\substack{d \text{ monic} \\ \deg(d) > [g/2]}} \frac{1}{|d|^2} \prod_{P|d} \frac{1}{|P|} \\ &= |D| \sum_{h > [g/2]} q^{-2h} \ll |D| q^{-g} \\ &\ll q^g. \end{aligned}$$

□

**Lemma 5.4.7.** We have that,

$$\begin{aligned} \frac{|D|}{\zeta_A(2)} \sum_{\substack{d \text{ monic} \\ \deg(d) \leq [g/2]}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} \left( \frac{q^{-\deg(d)}}{1-q^{-1}} \right) \\ = |D| \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left( 1 - \frac{1}{(|P|+1)|P|^2} \right) + O(q^g). \end{aligned} \quad (5.4.11)$$

*Proof.*

$$\begin{aligned} \frac{|D|}{\zeta_A(2)} \sum_{\substack{d \text{ monic} \\ \deg(d) \leq [g/2]}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} \left( \frac{q^{-\deg(d)}}{1-q^{-1}} \right) \\ = |D| \sum_{d \text{ monic}} \frac{\mu(d)}{|d|^2} \prod_{P|d} \frac{1}{|P|+1} - |D| \sum_{\substack{d \text{ monic} \\ \deg(d) > [g/2]}} \frac{\mu(d)}{|d|^2} \prod_{P|d} \frac{1}{|P|+1}. \end{aligned} \quad (5.4.12)$$

Writing the sum over all monic polynomials  $d$  as an Euler product and using the Lemma 5.4.6 in the sum over  $d$  such that  $\deg(d) > [g/2]$  we obtain the desired lemma. □

Using the same ideas used in the proof of Lemmas 5.4.6 and 5.4.7 we can also prove the following lemmas.

**Lemma 5.4.8.** *We have,*

1.

$$|D| \sum_{\substack{d \text{ monic} \\ \deg(d) > [g/2]}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} \ll q^{\frac{3}{2}g}. \quad (5.4.13)$$

2.

$$\frac{|D|q^{-g}q^{[(g-1)/2]+1}}{\zeta_A(2)(1-q)} \sum_{\substack{d \text{ monic} \\ \deg(d) > [(g-1)/2]}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} \ll q^g. \quad (5.4.14)$$

**Lemma 5.4.9.** *We have that,*

1.

$$\begin{aligned} & \frac{|D|}{\zeta_A(2)} \sum_{\substack{d \text{ monic} \\ \deg(d) \leq [g/2]}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} \left( \frac{q^{-[g/2]-1}}{1-q^{-1}} \right) \\ &= |D|q^{-[g/2]-1} \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left( 1 - \frac{1}{|P|(|P|+1)} \right) + O(q^g). \end{aligned} \quad (5.4.15)$$

2.

$$\begin{aligned} & \frac{|D|q^{-g}}{\zeta_A(2)} \sum_{\substack{d \text{ monic} \\ \deg(d) \leq [(g-1)/2]}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} \left( \frac{q^{[(g-1)/2]+1}}{1-q} \right) \\ &= \frac{|D|q^{-g}q^{[(g-1)/2]+1}}{\zeta_A(2)(1-q)} \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left( 1 - \frac{1}{|P|(|P|+1)} \right) + O(q^g). \end{aligned} \quad (5.4.16)$$

We present now our last lemma,

**Lemma 5.4.10.** *We have,*

$$\frac{|D|q^{-g}}{\zeta_A(2)(1-q)} \sum_{\substack{d \text{ monic} \\ \deg(d) \leq [(g-1)/2]}} \mu(d) \prod_{P|d} \frac{1}{|P|+1} \ll gq^g. \quad (5.4.17)$$

*Proof.* We have that,

$$\begin{aligned} \frac{|D|q^{-g}}{\zeta_A(2)(1-q)} \sum_{\substack{d \text{ monic} \\ \deg(d) \leq [(g-1)/2]}} \mu(d) \prod_{P|d} \frac{1}{|P|+1} &\ll |D|q^{-g} \sum_{\substack{d \text{ monic} \\ \deg(d) \leq [(g-1)/2]}} \frac{1}{|d|} \\ &\ll |D|q^{-g}([(g-1)/2]+1) \\ &\ll gq^g. \end{aligned}$$

□

*Proof of Proposition 5.4.1.* Our argument in this section follows closely the calculations presented in Chapter 3. From (5.4.6), our main goal is to obtain an asymptotic formula for

$$\begin{aligned} & \sum_{D \in \mathcal{H}_{2g+1,q}} L(1, \chi_D) \\ = & \sum_{D \in \mathcal{H}_{2g+1,q}} \sum_{n=0}^g \sum_{\substack{f \text{ monic} \\ \deg(f)=n}} \chi_D(f) q^{-n} + q^{-g} \sum_{D \in \mathcal{H}_{2g+1,q}} \sum_{m=0}^{g-1} \sum_{\substack{f \text{ monic} \\ \deg(f)=m}} \chi_D(f). \end{aligned} \quad (5.4.18)$$

We begin by establishing an asymptotic formula for the first term in the right-hand side of (5.4.18).

$$\begin{aligned} & \sum_{D \in \mathcal{H}_{2g+1,q}} \sum_{n=0}^g \sum_{\substack{f \text{ monic} \\ \deg(f)=n}} \chi_D(f) q^{-n} \\ = & \sum_{n=0}^g q^{-n} \sum_{D \in \mathcal{H}_{2g+1,q}} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f=l^2}} \chi_D(f) + \sum_{n=0}^g q^{-n} \sum_{D \in \mathcal{H}_{2g+1,q}} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f \neq \square}} \chi_D(f) \end{aligned} \quad (5.4.19)$$

Making use of the first part of Lemma 5.4.5 we can write (5.4.19) as

$$\sum_{D \in \mathcal{H}_{2g+1,q}} \sum_{n=0}^g \sum_{\substack{f \text{ monic} \\ \deg(f)=n}} \chi_D(f) q^{-n} = \sum_{n=0}^g q^{-n} \sum_{D \in \mathcal{H}_{2g+1,q}} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f=l^2}} \chi_D(f) + O((2q)^g). \quad (5.4.20)$$

For the square terms  $f = l^2$ , we can use Proposition 3.4.3 and we end up with

$$\begin{aligned} & \sum_{n=0}^g q^{-n} \sum_{D \in \mathcal{H}_{2g+1,q}} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f=l^2}} \chi_D(f) \\ = & \frac{|D|}{\zeta_A(2)} \sum_{m=0}^{\lfloor g/2 \rfloor} q^{-m} \sum_{\substack{d \text{ monic} \\ \deg(d) \leq m}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} + O(q^{g/2}) \\ = & \frac{|D|}{\zeta_A(2)} \sum_{\substack{d \text{ monic} \\ \deg(d) \leq \lfloor g/2 \rfloor}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} \left( \frac{(q^{-1})^{\deg(d)} - (q^{-1})^{\lfloor g/2 \rfloor + 1}}{1 - q^{-1}} \right) + O(q^{g/2}). \end{aligned}$$

$$\begin{aligned}
 &= \frac{|D|}{\zeta_A(2)} \sum_{\substack{d \text{ monic} \\ \deg(d) \leq [g/2]}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} \left( \frac{q^{-\deg(d)}}{1-q^{-1}} \right) \\
 &\quad - \frac{|D|}{\zeta_A(2)} \sum_{\substack{d \text{ monic} \\ \deg(d) \leq [g/2]}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} \left( \frac{q^{-[g/2]+1}}{1-q^{-1}} \right) + O(q^{g/2}). \quad (5.4.21)
 \end{aligned}$$

Now for the first term of (5.4.21) we can use Lemma 5.4.7 and for the second term we can use Lemma 5.4.9 and so we end up with the following formula for the square terms

$$\begin{aligned}
 \sum_{n=0}^g q^{-n} \sum_{D \in \mathcal{H}_{2g+1,q}} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f=l^2}} \chi_D(f) &= |D| \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left( 1 - \frac{1}{|P|^2(|P|+1)} \right) \\
 &\quad - |D| q^{-[g/2]-1} \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left( 1 - \frac{1}{|P|(|P|+1)} \right) + O(q^g). \quad (5.4.22)
 \end{aligned}$$

Substituting (5.4.22) in (5.4.20) we have that,

$$\begin{aligned}
 \sum_{D \in \mathcal{H}_{2g+1,q}} \sum_{n=0}^g \sum_{\substack{f \text{ monic} \\ \deg(f)=n}} \chi_D(f) q^{-n} &= |D| \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left( 1 - \frac{1}{|P|^2(|P|+1)} \right) \\
 &\quad - |D| q^{-[g/2]-1} \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left( 1 - \frac{1}{|P|(|P|+1)} \right) + O((2q)^g). \quad (5.4.23)
 \end{aligned}$$

For the second term in the right-hand side of (5.4.18) we will mimic the calculations above to end up with,

$$\begin{aligned}
 &q^{-g} \sum_{D \in \mathcal{H}_{2g+1,q}} \sum_{m=0}^{g-1} \sum_{\substack{f \text{ monic} \\ \deg(f)=m}} \chi_D(f) \\
 &= \frac{|D| q^{-g}}{\zeta_A(2)} \sum_{\substack{d \text{ monic} \\ \deg(d) \leq [(g-1)/2]}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} \sum_{\deg(d) \leq n \leq [(g-1)/2]} q^n + O(q^{[(g-1)/2]}) \\
 &+ O((2q)^g)
 \end{aligned}$$

$$\begin{aligned}
&= \frac{|D|q^{-g}}{\zeta_A(2)} \sum_{\substack{d \text{ monic} \\ \deg(d) \leq [(g-1)/2]}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} \left( \frac{q^{\deg(d)}}{1-q} \right) \\
&- \frac{|D|q^{-g}}{\zeta_A(2)} \sum_{\substack{d \text{ monic} \\ \deg(d) \leq [(g-1)/2]}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} \left( \frac{q^{[(g-1)/2]+1}}{1-q} \right) + O((2q)^g), \quad (5.4.24)
\end{aligned}$$

where the error  $O((2q)^g)$  arises when we consider  $f \neq \square$  and using part (2) of Lemma 5.4.5.

For the first term in (5.4.24) we use the bound given in Lemma 5.4.10 and for the second term we have,

$$\begin{aligned}
&\frac{|D|q^{-g}}{\zeta_A(2)} \sum_{\substack{d \text{ monic} \\ \deg(d) \leq [(g-1)/2]}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} \left( \frac{q^{[(g-1)/2]+1}}{1-q} \right) \\
&= \frac{|D|q^{-g}}{\zeta_A(2)} \left( \sum_{d \text{ monic}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} \left( \frac{q^{[(g-1)/2]+1}}{1-q} \right) \right) \\
&\quad - \frac{|D|q^{-g}}{\zeta_A(2)} \left( \sum_{\substack{d \text{ monic} \\ \deg(d) > [(g-1)/2]}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} \left( \frac{q^{[(g-1)/2]+1}}{1-q} \right) \right). \quad (5.4.25)
\end{aligned}$$

And we can use part (2) of the Lemma 5.4.9, and so we have that,

$$\begin{aligned}
&\frac{|D|q^{-g}}{\zeta_A(2)} \sum_{\substack{d \text{ monic} \\ \deg(d) \leq [(g-1)/2]}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} \left( \frac{q^{[(g-1)/2]+1}}{1-q} \right) \\
&= \frac{|D|q^{-g}q^{[(g-1)/2]+1}}{\zeta_A(2)(1-q)} \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left( 1 - \frac{1}{|P|(|P|+1)} \right) + O((2q)^g). \quad (5.4.26)
\end{aligned}$$

So, we can conclude that,

$$\begin{aligned}
&q^{-g} \sum_{D \in \mathcal{H}_{2g+1,q}} \sum_{m=0}^{g-1} \sum_{\substack{f \text{ monic} \\ \deg(f)=m}} \chi_D(f) \\
&= - \frac{|D|q^{-g}q^{[(g-1)/2]+1}}{\zeta_A(2)(1-q)} \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left( 1 - \frac{1}{|P|(|P|+1)} \right) + O((2q)^g). \quad (5.4.27)
\end{aligned}$$

Putting together the equations (5.4.23) and (5.4.27) and factoring  $|D|$  we have that the proof is complete.  $\square$

### 5.4.2 Proof of Proposition 5.4.3

Remember that the  $L$ -function associated to the character  $\chi_P(f)$ , where  $P \in A$  is a monic irreducible polynomial of degree odd, is the same as the  $L$ -polynomial of the zeta function associated to the curve  $C_P : y^2 = P(T)$ , where  $\deg(P) = 2M + 1$ . Therefore, we have the following representation for  $L(s, \chi_P)$

$$L(s, \chi_P) = \sum_{\substack{f_1 \text{ monic} \\ \deg(f_1) \leq M}} \frac{\chi_P(f_1)}{|f_1|^s} + (q^{1-2s})^M \sum_{\substack{f_2 \text{ monic} \\ \deg(f_2) \leq M-1}} \frac{\chi_P(f_2)}{|f_2|^{1-s}}. \quad (5.4.28)$$

#### Preliminary Lemmas

Let us now introduce some preliminary lemmas which are similar to those proven in Chapter 4. Note that the main tool used in the following pages is the Polynomial Prime Number Theorem, which says that

$$\pi_A(n) = \frac{q^n}{n} + O(q^{n/2}), \quad (5.4.29)$$

where  $\pi_A(n)$  is the number of monic irreducible polynomials of degree  $n$ . Now we are in a position to present our first auxiliary lemma.

**Lemma 5.4.11.** *We have that,*

$$\begin{aligned}
 & \sum_{\substack{n=0 \\ 2|n}}^M q^{-n} \sum_{\substack{l \text{ monic} \\ \deg(l)=n/2}} \sum_{\substack{P \text{ monic} \\ \deg(P)=2M+1 \\ (P,l)=1}} 1 \\
 &= \frac{q^{2M+1}}{2M+1} \left( \frac{q^{-[M/2]} (-1 + q^{1+[M/2]})}{-1 + q} \right) - \frac{1}{2M+1} \left( \frac{q^{-[M/2]} (-1 + q^{1+[M/2]})}{-1 + q} \right) \\
 &+ O(q^M).
 \end{aligned} \tag{5.4.30}$$

*Proof.* We start by writing

$$\begin{aligned}
 & \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1}} \sum_{n=0}^M \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f=l^2}} \chi_P(f) q^{-n} = \sum_{n=0}^M q^{-n} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f=l^2}} \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1}} \chi_P(f) \\
 &= \sum_{\substack{n=0 \\ 2|n}}^M q^{-n} \sum_{\substack{l \text{ monic} \\ \deg(l)=n/2}} \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1}} \chi_P(l^2) = \sum_{\substack{n=0 \\ 2|n}}^M q^{-n} \sum_{\substack{l \text{ monic} \\ \deg(l)=n/2}} \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1 \\ (P,l)=1}} 1 \\
 &= \sum_{\substack{n=0 \\ 2|n}}^M q^{-n} \sum_{\substack{l \text{ monic} \\ \deg(l)=n/2}} \left( \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1}} 1 - \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1 \\ P|l}} 1 \right),
 \end{aligned} \tag{5.4.31}$$

and so we can write the above quantity as

$$\begin{aligned}
 & \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1}} \sum_{n=0}^M \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f=l^2}} \chi_P(f) q^{-n} \\
 &= \sum_{\substack{n=0 \\ 2|n}}^M q^{-n} \sum_{\substack{l \text{ monic} \\ \deg(l)=n/2}} \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1}} 1 - \sum_{\substack{n=0 \\ 2|n}}^M q^{-n} \sum_{\substack{l \text{ monic} \\ \deg(l)=n/2}} \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1 \\ P|l}} 1,
 \end{aligned} \tag{5.4.32}$$

and making use of the Polynomial Prime Number Theorem 2.1.5 we obtain



that (5.4.32) is

$$\begin{aligned}
 &= \sum_{\substack{n=0 \\ 2|n}}^M q^{-n} \sum_{\substack{l \text{ monic} \\ \deg(l)=n/2}} \left( \frac{q^{2M+1}}{2M+1} + O\left(q^{\frac{2M+1}{2}}\right) \right) \\
 &\quad - \sum_{\substack{n=0 \\ 2|n}}^M q^{-n} \sum_{\substack{P \text{ monic} \\ \deg(P)=2M+1}} \sum_{\substack{l \text{ monic} \\ \deg(l)=n/2 \\ P|l}} 1 \\
 &= \sum_{\substack{n=0 \\ 2|n}}^M q^{-n} \sum_{\substack{l \text{ monic} \\ \deg(l)=n/2}} \frac{q^{2M+1}}{2M+1} - \sum_{\substack{n=0 \\ 2|n}}^M q^{-n} \sum_{\substack{P \text{ monic} \\ \deg(P)=2M+1}} \sum_{\substack{a \text{ monic} \\ \deg(a)=\frac{n}{2}=-2M-1}} 1 \\
 &+ O\left( \sum_{\substack{n=0 \\ 2|n}}^M q^{-n} \sum_{\substack{l \text{ monic} \\ \deg(l)=n/2}} q^{\frac{2M+1}{2}} \right) \\
 &= \frac{q^{2M+1}}{2M+1} \sum_{\substack{n=0 \\ 2|n}}^M q^{-n/2} - \sum_{\substack{n=0 \\ 2|n}}^M q^{-n/2} q^{-2M-1} \sum_{\substack{P \text{ monic} \\ \deg(P)=2M+1}} 1 + O\left( q^{\frac{2M+1}{2}} \sum_{\substack{n=0 \\ 2|n}}^M q^{-n/2} \right) \\
 &= \frac{q^{2M+1}}{2M+1} \sum_{n=0}^{\lfloor \frac{M}{2} \rfloor} q^{-n} - q^{-2M-1} \sum_{n=0}^{\lfloor \frac{M}{2} \rfloor} q^{-n} \left( \frac{q^{2M+1}}{2M+1} + O\left(q^{\frac{2M+1}{2}}\right) \right) \\
 &+ O\left( q^M \sum_{n=0}^{\lfloor \frac{M}{2} \rfloor} q^{-n} \right) \\
 &= \frac{q^{2M+1}}{2M+1} \left( \frac{q^{-[M/2](-1+q^{1+[M/2]})}}{-1+q} \right) - \frac{1}{2M+1} \left( \frac{q^{-[M/2](-1+q^{1+[M/2]})}}{-1+q} \right) \\
 &+ O(q^M) + O\left( q^{-M} q^{-[M/2](-1+q^{1+[M/2]})} \right).
 \end{aligned}$$

(5.4.33)

which proves the proposition.  $\square$

**Lemma 5.4.12.** *We have that,*

$$\sum_{n=0}^M q^{-n} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f \neq \square}} \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1}} \chi_P(f) = O\left(\frac{q^M M^2}{2M+1}\right). \quad (5.4.34)$$

*Proof.* Using Theorem 4.3.2 we have that,

$$\begin{aligned} \sum_{n=0}^M q^{-n} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f \neq \square}} \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1}} \chi_P(f) &\ll \sum_{n=0}^M q^{-n} \sum_{\substack{f \text{ monic} \\ \deg(f)=n}} \frac{n}{2M+1} q^M \\ &= \frac{q^M}{2M+1} \sum_{n=0}^M n \\ &= \frac{q^M (M^2 + M)}{2M+1} \\ &\ll \frac{q^M M^2}{2M+1}. \end{aligned} \quad (5.4.35)$$

□

**Lemma 5.4.13.** *We have that,*

$$\begin{aligned} &\sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1}} \sum_{n=0}^M \sum_{\substack{f \text{ monic} \\ \deg(f)=n}} \chi_P(f) q^{-n} \\ &= \frac{q^{2M+1}}{2M+1} \left( \frac{q^{-[M/2]} (-1 + q^{1+[M/2]})}{-1 + q} \right) - \frac{1}{2M+1} \left( \frac{q^{-[M/2]} (-1 + q^{1+[M/2]})}{-1 + q} \right) \\ &+ O\left(\frac{q^M M^2}{2M+1}\right). \end{aligned} \quad (5.4.36)$$

*Proof.* Follows directly from Lemmas 5.4.11 and 5.4.12. □

**Lemma 5.4.14.** *We have that,*

$$\begin{aligned} &q^{-M} \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1}} \sum_{n=0}^{M-1} \sum_{\substack{l \text{ monic} \\ 2|n \\ \deg(l)=n/2}} \chi_P(l^2) \\ &= \frac{q^{M+1}}{2M+1} \left( \frac{-1 + q^{1+\lceil \frac{M-1}{2} \rceil}}{-1 + q} \right) - \frac{q^{-M}}{2M+1} \left( \frac{-1 + q^{1+\lceil \frac{M-1}{2} \rceil}}{-1 + q} \right) + O\left(q^{\lceil \frac{M-1}{2} \rceil}\right). \end{aligned} \quad (5.4.37)$$

*Proof.* We can write

$$\begin{aligned}
 & q^{-M} \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1}} \sum_{n=0}^{M-1} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f=l^2}} \chi_P(f) \\
 &= q^{-M} \sum_{n=0}^{M-1} \sum_{\substack{l \text{ monic} \\ 2|n \\ \deg(l)=n/2}} \left( \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1}} 1 - \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1 \\ P|l}} 1 \right) \\
 &= \frac{q^{M+1}}{2M+1} \sum_{n=0}^{\lfloor \frac{M-1}{2} \rfloor} q^n - \frac{q^{-M}}{2M+1} \sum_{n=0}^{\lfloor \frac{M-1}{2} \rfloor} q^n + O\left(q^{1+\lfloor \frac{M-1}{2} \rfloor}\right) \\
 &+ O\left(q^{-2M} \sum_{n=0}^{\lfloor \frac{M-1}{2} \rfloor} q^n\right) \\
 &= \frac{q^{M+1}}{2M+1} \left( \frac{-1+q^{1+\lfloor \frac{M-1}{2} \rfloor}}{-1+q} \right) - \frac{q^{-M}}{2M+1} \left( \frac{-1+q^{1+\lfloor \frac{M-1}{2} \rfloor}}{-1+q} \right) + O\left(q^{\lfloor \frac{M-1}{2} \rfloor}\right).
 \end{aligned} \tag{5.4.38}$$

□

**Lemma 5.4.15.** *We have that,*

$$q^{-M} \sum_{n=0}^{M-1} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f \neq \square}} \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1}} \chi_P(f) = O\left(\frac{q^M M}{2M+1}\right). \tag{5.4.39}$$

*Proof.* Using the same arguments of Lemma 5.4.12 we have that,

$$\begin{aligned}
 q^{-M} \sum_{n=0}^{M-1} \sum_{\substack{f \text{ monic} \\ \deg(f)=n \\ f \neq \square}} \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1}} \chi_P(f) &\ll \sum_{n=0}^{M-1} \sum_{\substack{f \text{ monic} \\ \deg(f)=n}} \frac{n}{2M+1} \\
 &= \frac{1}{2M+1} \left( \frac{q - Mq^m - q^{1+M} + Mq^{1+M}}{(-1+q)^2} \right) \\
 &\ll \frac{1}{2M+1} Mq^M.
 \end{aligned} \tag{5.4.40}$$

□

**Lemma 5.4.16.** *We have that,*

$$\begin{aligned}
 & \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=2M+1}} q^{-M} \sum_{\substack{f \text{ monic} \\ \deg(f) \leq (M-1)}} \chi_P(f) \\
 &= \frac{q^{M+1}}{2M+1} \left( \frac{-1 + q^{1+\left[\frac{M-1}{2}\right]}}{-1 + q} \right) - \frac{q^{-M}}{2M+1} \left( \frac{-1 + q^{1+\left[\frac{M-1}{2}\right]}}{-1 + q} \right) \\
 &+ O\left(\frac{q^M M}{2M+1}\right).
 \end{aligned} \tag{5.4.41}$$

*Proof.* Follows directly from Lemmas 5.4.14 and 5.4.15.  $\square$

*Proof of Proposition 5.4.3.* Putting the Lemmas 5.4.13 and 5.4.16 together and after some simple algebraic manipulations, we have the proof of the proposition as desired.  $\square$

## 5.5 Proof of the Main Theorems

One last ingredient is need for we be able to present a complete proof of the main theorems of this chapter, this ingredient is the following result due to Artin [Artin].

**Theorem 5.5.1** (Artin). *Let  $D \in A$  be a square-free polynomial of degree  $M$ . Then if  $M$  is odd, we have that*

$$L(1, \chi_D) = \frac{\sqrt{q}}{\sqrt{|D|}} h_D. \tag{5.5.1}$$

Now we can present the proof of the main theorems.

*Proof of Theorem 5.3.1.* Using Corollary 5.4.2 and the Artin's result (5.5.1), the desired formula follows.  $\square$

*Proof of Theorem 5.3.2.* Using Corollary 5.4.4 and Artin's result (5.5.1) with  $D = P$ , the desired formula follows.  $\square$

# Chapter 6

## Integral Moments of $L$ -functions in Function Fields

In Chapter 1, we mentioned the Keating and Snaith conjectures about the moments of  $L$ -functions. In particular, they conjectured (see Conjecture 1.5.8) a formula for the leading asymptotic of

$$\sum_{0 < d \leq D}^* L(\tfrac{1}{2}, \chi_d)^k, \quad \text{for } \Re(k) \geq 0, \quad (6.0.1)$$

as  $D \rightarrow \infty$ . However, to gain a full understanding of the structure of moments of  $L$ -functions we need to go beyond the leading order asymptotic and determine the principal lower order terms and, if possible, study the profound structure of the error-term. The main idea of this Chapter is to present and extend to the function field setting the heuristic, previously developed, by Conrey, Farmer, Keating, Rubinstein and Snaith [CFKRS05], for the integral moments of  $L$ -functions defined over number fields. Specifically, we give a heuristic for the moments of a family of  $L$ -functions associated with hyperelliptic curves of genus  $g$  over a fixed finite field  $\mathbb{F}_q$  in the limit as  $g \rightarrow \infty$ , we present the function field analogue of the Conjecture 1.5.3 presented in [CFKRS05]. To accomplish this task we will adapt for the function field setting the recipe for conjecturing moments of  $L$ -functions.

Like in the number field case, our conjecture has a striking resemblance

to the corresponding formulae for the moments of characteristic polynomials of random matrices, since  $L(s, \chi_D)$  has a spectral interpretation and therefore Random Matrix Theory should be a good model.

**Remark 6.0.2.** *Os calculos e resultados apresentados neste capitulo e no proximo capitulo tambem aparecem em [And-Kea12].*

## 6.1 Conjecture for Integral Moments of $L(\frac{1}{2}, \chi_D)$ over the rational function field

Under the philosophy that statistical properties of the zeros of  $L$ -functions and the eigenvalue distributions of random unitary matrices are connected in some way, Conrey *et.al.* in [CFKRS05, CFKRS08] applying number-theoretic heuristics derived an asymptotic expansion for the  $k^{\text{th}}$  integral moment of  $L(\frac{1}{2}, \chi_d)$ , which includes the lower order terms. We present their conjecture and method, so we can analyze and observe the similarities between the classical case and the function field case that we develop in this chapter.

**Conjecture 6.1.1.** *[Conrey, Farmer, Keating, Rubinstein and Snaith] Suppose  $g(u)$  is a suitable weight function with support in either  $(0, \infty)$  or  $(-\infty, 0)$ , and let  $X_d(s) = |d|^{\frac{1}{2}-s} X(s, a)$  where  $a = 0$  if  $d > 0$  and  $a = 1$  if  $d < 0$ , and*

$$X(s, a) = \pi^{s-1/2} \Gamma\left(\frac{1+a-s}{2}\right) / \Gamma\left(\frac{s+a}{2}\right). \quad (6.1.1)$$

*That is,  $X_d(s)$  is the factor in the functional equation*

$$L(s, \chi_d) = \varepsilon_d X_d(s) L(1-s, \chi_d).$$

*Summing over fundamental discriminants  $d$  we have*

$$\sum_d^* L(\frac{1}{2}, \chi_d)^k g(|d|) = \sum_d^* Q_k(\log |d|) (1 + O(|d|^{-\frac{1}{2}+\varepsilon})) g(|d|) \quad (6.1.2)$$

*where  $Q_k$  is the polynomial of degree  $k(k+1)/2$  given by the  $k$ -fold residue*

$$Q_k(x) = \frac{(-1)^{k(k-1)/2} 2^k}{k!} \frac{1}{(2\pi i)^k} \oint \dots \oint \frac{G(z_1, \dots, z_k) \Delta(z_1^2, \dots, z_k^2)^2}{\prod_{j=1}^k z_j^{2k-1}} \times e^{\frac{x}{2} \sum_{j=1}^k z_j} dz_1 \dots dz_k, \quad (6.1.3)$$

6.1. Conjecture for Integral Moments of  $L(\frac{1}{2}, \chi_D)$  over the rational function field

---

with

$$G(z_1, \dots, z_k) = A_k(z_1, \dots, z_k) \prod_{j=1}^k X(\frac{1}{2} + z_j, a)^{-\frac{1}{2}} \prod_{1 \leq i < j \leq k} \zeta(1 + z_i + z_j), \quad (6.1.4)$$

$\Delta(z_1, \dots, z_k)$  the Vandermonde determinant given by

$$\Delta(z_1, \dots, z_k) = \prod_{1 \leq i < j \leq k} (z_j - z_i), \quad (6.1.5)$$

and  $A_k$  is the Euler product, absolutely convergent for  $|\Re z_j| < \frac{1}{2}$ , defined by

$$\begin{aligned} A_k(z_1, \dots, z_k) &= \prod_p \prod_{1 \leq i < j \leq k} \left(1 - \frac{1}{p^{1+z_i+z_j}}\right) \\ &\times \left(\frac{1}{2} \left(\prod_{j=1}^k \left(1 - \frac{1}{p^{\frac{1}{2}+z_j}}\right)^{-1} + \prod_{j=1}^k \left(1 + \frac{1}{p^{\frac{1}{2}+z_j}}\right)^{-1}\right) + \frac{1}{p}\right) \\ &\times \left(1 + \frac{1}{p}\right)^{-1}. \end{aligned} \quad (6.1.6)$$

More generally, if  $\mathcal{F}$  is the family of real primitive Dirichlet  $L$ -functions then

$$S_k(\mathcal{F}, \alpha, g) = \sum_d^* Q_k(\log |d|, \alpha) (1 + O(|d|^{-\frac{1}{2}+\epsilon})) g(|d|), \quad (6.1.7)$$

where

$$\begin{aligned} Q_k(x, \alpha) &= \frac{(-1)^{k(k-1)/2} 2^k}{k!} \frac{1}{(2\pi i)^k} \\ &\times \oint \cdots \oint \frac{G(z_1, \dots, z_k) \Delta(z_1^2, \dots, z_k^2)^2 \prod_{j=1}^k z_j}{\prod_{i=1}^k \prod_{j=1}^k (z_j - \alpha_i)(z_j + \alpha_i)} \\ &\times e^{\frac{x}{2} \sum_{j=1}^k z_j} dz_1 \cdots dz_k, \end{aligned} \quad (6.1.8)$$

where the path of integration encloses the  $\pm \alpha$ 's.

This same conjecture can be obtained through the use of multiple Dirichlet series techniques [Diac-Gold-Hoff]. The formulae presented in the Conjecture 6.1.1 matches the corresponding formulae (which are identities) of the theorem presented below quoted from [CFKRS03] for characteristic polynomials of random matrices. We can also compare the theorem with the Conjecture 6.1.3 which we present in this chapter and, again we see the same similarities.

**Theorem 6.1.2.** *Let,*

$$J_k(USp(2N), \alpha) = \int_{USp(2N)} \Lambda_M(e^{-\alpha_1}) \dots \Lambda_M(e^{-\alpha_k}) dA. \quad (6.1.9)$$

*Then,*

$$J_k(USp(2N), 0) = \left( 2^{k(k+1)/2} \prod_{j=1}^k \frac{j!}{(2j)!} \right) \prod_{1 \leq i \leq j \leq k} \left( N + \frac{i+j}{2} \right), \quad (6.1.10)$$

where  $A$  is in the group of symplectic unitary matrices,  $USp(2N)$  and

$$\Lambda_A(s) = \det(I - As).$$

More generally, with

$$G(z_1, \dots, z_k) = \prod_{1 \leq i \leq j \leq k} (1 - e^{-z_i - z_j})^{-1}$$

we have

$$\begin{aligned} J_k(USp(2N), \alpha) &= \frac{(-1)^{k(k-1)/2} 2^k}{k!} \frac{1}{(2\pi i)^k} \\ &\times \oint \dots \oint \frac{G(z_1, \dots, z_k) \Delta(z_1^2, \dots, z_k^2)^2 \prod_{j=1}^k z_j}{\prod_{i=1}^k \prod_{j=1}^k (z_j - \alpha_i)(z_j + \alpha_i)} \\ &\times e^{N \sum_{j=1}^k z_j} dz_1 \dots dz_k, \end{aligned} \quad (6.1.11)$$

where the contours of integration enclose the  $\pm\alpha$ 's.

By applying an adapted version of the recipe used to write the general conjectures for moments of  $L$ -functions we obtain the following conjecture for moments of quadratic Dirichlet  $L$ -functions over function fields

**Conjecture 6.1.3.** *Suppose that  $q \equiv 1 \pmod{4}$  is the fixed cardinality of the finite field  $\mathbb{F}_q$  and let  $\mathcal{X}_D(s) = |D|^{1/2-s} \mathcal{X}(s)$  and*

$$\mathcal{X}(s) = q^{-1/2+s}. \quad (6.1.12)$$

*That is,  $\mathcal{X}_D(s)$  is the factor in the functional equation*

$$L(s, \chi_D) = \mathcal{X}_D(s) L(1-s, \chi_D). \quad (6.1.13)$$



6.1. Conjecture for Integral Moments of  $L(\frac{1}{2}, \chi_D)$  over the rational function field

---

Summing over fundamental discriminants  $D \in \mathcal{H}_{2g+1, q}$  we have

$$\sum_{D \in \mathcal{H}_{2g+1, q}} L(\frac{1}{2}, \chi_D)^k = \sum_{D \in \mathcal{H}_{2g+1, q}} Q_k(\log_q |D|)(1 + O(|D|^{-\frac{1}{2}+\epsilon})) \quad (6.1.14)$$

where  $Q_k$  is the polynomial of degree  $k(k+1)/2$  given by the  $k$ -fold residue

$$Q_k(x) = \frac{(-1)^{k(k-1)/2} 2^k}{k!} \frac{1}{(2\pi i)^k} \oint \cdots \oint \frac{G(z_1, \dots, z_k) \Delta(z_1^2, \dots, z_k^2)^2}{\prod_{j=1}^k z_j^{2k-1}} \times q^{\frac{x}{2} \sum_{j=1}^k z_j} dz_1 \cdots dz_k, \quad (6.1.15)$$

where  $\Delta(z_1, \dots, z_k)$  is defined as in (6.1.5),

$$G(z_1, \dots, z_k) = A_k(z_1, \dots, z_k) \prod_{j=1}^k \mathcal{X}(\frac{1}{2} + z_j)^{-\frac{1}{2}} \prod_{1 \leq i < j \leq k} \zeta_A(1 + z_i + z_j), \quad (6.1.16)$$

and  $A_k$  is the Euler product, absolutely convergent for  $|\Re(z_j)| < \frac{1}{2}$ , defined by

$$A_k(z_1, \dots, z_k) = \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \prod_{1 \leq i < j \leq k} \left( 1 - \frac{1}{|P|^{1+z_i+z_j}} \right) \times \left( \frac{1}{2} \left( \prod_{j=1}^k \left( 1 - \frac{1}{|P|^{\frac{1}{2}+z_j}} \right)^{-1} + \prod_{j=1}^k \left( 1 + \frac{1}{|P|^{\frac{1}{2}+z_j}} \right)^{-1} \right) + \frac{1}{|P|} \right) \times \left( 1 + \frac{1}{|P|} \right)^{-1}. \quad (6.1.17)$$

More generally, we have

$$\begin{aligned} & \sum_{D \in \mathcal{H}_{2g+1, q}} L(\frac{1}{2} + \alpha_1, \chi_D) \cdots L(\frac{1}{2} + \alpha_k, \chi_D) \\ &= \sum_{D \in \mathcal{H}_{2g+1, q}} \prod_{j=1}^k \mathcal{X}(\frac{1}{2} + \alpha_j)^{-\frac{1}{2}} |D|^{-\frac{1}{2} \sum_{j=1}^k \alpha_j} Q_k(\log_q |D|, \alpha) (1 + O(|D|^{-\frac{1}{2}+\epsilon})), \end{aligned} \quad (6.1.18)$$

where

$$Q_k(x, \alpha) = \frac{(-1)^{k(k-1)/2} 2^k}{k!} \frac{1}{(2\pi i)^k} \times \oint \cdots \oint \frac{G(z_1, \dots, z_k) \Delta(z_1^2, \dots, z_k^2)^2 \prod_{j=1}^k z_j}{\prod_{i=1}^k \prod_{j=1}^k (z_j - \alpha_i)(z_j + \alpha_i)} \times q^{\frac{x}{2} \sum_{j=1}^k z_j} dz_1 \cdots dz_k, \quad (6.1.19)$$

and the path of integration encloses the  $\pm \alpha$ 's.

## 6.2 Heuristic Derivation of the Conjecture

In this section we will present the details of the recipe for conjecturing moments of  $L$ -functions associated with hyperelliptic curves of genus  $g$  over a fixed finite field  $\mathbb{F}_q$  as  $g \rightarrow \infty$ . To do this, we will adapt to the function field setting the recipe presented in [CFKRS05]. Note that the recipe is used without rigorous justification in each of its steps, but when seen as a whole it serves to produce a conjecture for the moments of  $L$ -functions that is consistent with its random matrix analogues and with all results known to date.

Let  $D \in \mathcal{H}_{2g+1,q}$ . For a fixed  $k$ , we seek an asymptotic expression for

$$\sum_{D \in \mathcal{H}_{2g+1,q}} L\left(\frac{1}{2}, \chi_D\right)^k, \quad (6.2.1)$$

as  $g \rightarrow \infty$ . To achieve this we consider the more general expression obtained by introducing small shifts, say  $\alpha_1, \dots, \alpha_k$

$$\sum_{D \in \mathcal{H}_{2g+1,q}} L\left(\frac{1}{2} + \alpha_1, \chi_D\right) \dots L\left(\frac{1}{2} + \alpha_k, \chi_D\right). \quad (6.2.2)$$

By introducing such shifts, hidden structures are revealed in the form of symmetries and the calculations are simplified by the removal of higher order poles. In the end we let each  $\alpha_1, \dots, \alpha_k$  tend to 0 to recover (6.2.1).

### 6.2.1 Some Analogies Between Classical $L$ -functions and $L$ -functions over Function Fields

The starting point to conjecture moments for  $L$ -functions is the use of the approximate functional equation. For the hyperelliptic ensemble considered here, the analogue of the approximate functional equation is given by

$$L(s, \chi_D) = \sum_{\substack{n \text{ monic} \\ \deg(n) \leq g}} \frac{\chi_D(n)}{|n|^s} + \mathcal{X}_D(s) \sum_{\substack{n \text{ monic} \\ \deg(n) \leq g-1}} \frac{\chi_D(n)}{|n|^{1-s}}, \quad (6.2.3)$$

where  $D \in \mathcal{H}_{2g+1,q}$  and  $\mathcal{X}_D(s) = q^{g(1-2s)}$ . Note that we can write,

$$\mathcal{X}_D(s) = |D|^{\frac{1}{2}-s} \mathcal{X}(s), \quad (6.2.4)$$

where  $\mathcal{X}(s) = q^{-\frac{1}{2}+s}$  corresponds to the gamma factor that appears in the classical quadratic  $L$ -functions. Now we will present the following simple lemma which will be used in the recipe and which make the analogy between the function field case and the number field case more direct.

**Lemma 6.2.1.** *We have that,*

$$\mathcal{X}_D(s)^{1/2} = \mathcal{X}_D(1-s)^{-1/2}, \quad (6.2.5)$$

and

$$\mathcal{X}_D(s)\mathcal{X}_D(1-s) = 1. \quad (6.2.6)$$

*Proof.* The proof is straightforward and follows directly from the definition of  $\mathcal{X}_D(s)$ . □

It is convenient to remember that as in the classical case, we have that the following formula holds for  $L$ -functions in function fields

$$L(s, \chi_D) = \mathcal{X}_D(s)L(1-s, \chi_D). \quad (6.2.7)$$

This is a rather pretentious way of writing the functional equation for the  $L$ -polynomial of the zeta function associated to the curve  $C_D$ .

## 6.2.2 Applying the Recipe for $L$ -Functions over Function Fields

For ease of presentation, we will work with a slightly different  $L$ -function. Namely, we consider the  $Z$ -function

$$Z_{\mathcal{L}}(s, \chi_D) = \mathcal{X}_D(s)^{-1/2}L(s, \chi_D), \quad (6.2.8)$$

which satisfies a more symmetric functional equation, in this case

**Lemma 6.2.2.** *The function  $Z_{\mathcal{L}}(s, \chi_D)$  satisfies the following functional equation,*

$$Z_{\mathcal{L}}(s, \chi_D) = Z_{\mathcal{L}}(1-s, \chi_D). \quad (6.2.9)$$

*Proof.* This follows from a direct application of the first equation of Lemma 6.2.1.  $\square$

Thus, we would like to produce an asymptotic for the  $k$ -shifted moment

$$L_D(s) = \sum_{D \in \mathcal{H}_{2g+1,q}} Z(s; \alpha_1, \dots, \alpha_k), \quad (6.2.10)$$

where

$$Z(s; \alpha_1, \dots, \alpha_k) = \prod_{j=1}^k Z_{\mathcal{L}}(s + \alpha_j, \chi_D). \quad (6.2.11)$$

Making use of (6.2.3) and Lemma 6.2.1 we have that

$$Z_{\mathcal{L}}(s, \chi_D) = \mathcal{X}_D(s)^{-1/2} \sum_{\substack{n \text{ monic} \\ \deg(n) \leq g}} \frac{\chi_D(n)}{|n|^s} + \mathcal{X}_D(1-s)^{-1/2} \sum_{\substack{m \text{ monic} \\ \deg(m) \leq g-1}} \frac{\chi_D(m)}{|m|^{1-s}}. \quad (6.2.12)$$

As a matter of completeness we present the general recipe extract from Conrey *et.al* [CFKRS05] and we will follow their recipe making adjustments for function fields when necessary.

Suppose  $\mathcal{L}$  is an  $L$ -function and  $f$  is a character with conductor  $c(f)$ , as described in Section 3 of [CFKRS05]. So we have

$$Z_{\mathcal{L}}(s, f) = \varepsilon_f^{-\frac{1}{2}} \mathcal{X}_f(s)^{-\frac{1}{2}} \mathcal{L}(s, f), \quad (6.2.13)$$

which satisfies the functional equation

$$Z_{\mathcal{L}}(s, f) = \overline{Z_{\mathcal{L}}(1-s, f)}, \quad (6.2.14)$$

so  $Z_{\mathcal{L}}(s, f)$  is real on the  $\frac{1}{2}$ -line. Note that  $\varepsilon_f^{-1/2}$  involves a choice which must be made consistently.

We consider the moment

$$\sum_{f \in \mathcal{F}} Z_{\mathcal{L}}(\frac{1}{2} + \alpha_1, f) \dots Z_{\mathcal{L}}(\frac{1}{2} + \alpha_k, f) g(c(f)) \quad (6.2.15)$$

where  $g$  is a suitable test function.

Here is the recipe for conjecturing a formula for the above moment:

1. Start with a product of  $k$  shifted  $L$ -functions:

$$Z_f(s, \alpha_1, \dots, \alpha_k) = Z_{\mathcal{L}}(s + \alpha_1, f) \dots Z_{\mathcal{L}}(s + \alpha_k, f). \quad (6.2.16)$$

2. Replace each  $L$ -function with the two terms from its approximate functional equation, ignoring the remainder term. Multiply out the resulting expression to obtain  $2^k$  terms. Write those terms as

$$(\text{product of } \varepsilon_f \text{ factors})(\text{product of } \mathcal{X}_f \text{ factors}) \sum_{n_1, \dots, n_k} (\text{summand}). \quad (6.2.17)$$

3. Replace each product of  $\varepsilon_f$ -factors by its expected value when averaged over the family.
4. Replace each summand by its expected value when averaged over the family.
5. Extend each of  $n_1, \dots, n_k$  to all positive integers and call the total of  $M(s, \alpha_1, \dots, \alpha_{2k})$ .
6. The conjecture is

$$\begin{aligned} & \sum_{f \in \mathcal{F}} Z_f\left(\frac{1}{2}, \alpha_1, \dots, \alpha_{2k}\right) g(c(f)) \\ &= \sum_{f \in \mathcal{F}} M_f\left(\frac{1}{2}, \alpha_1, \dots, \alpha_{2k}\right) (1 + O(e^{(-\frac{1}{2} + \varepsilon)c(f)})) g(c(f)), \end{aligned} \quad (6.2.18)$$

for all  $\varepsilon > 0$ , where  $g$  is a suitable weight function.

Let us now exhibit the technical details involved in each of these steps when we adapt it for  $L$ -functions associated with hyperelliptic curves over finite fields.

- (1) **We start with a product of  $k$  shifted  $L$ -functions:**

$$Z(s; \alpha_1, \dots, \alpha_k) = Z_{\mathcal{L}}(s + \alpha_1, \chi_D) \dots Z_{\mathcal{L}}(s + \alpha_k, \chi_D). \quad (6.2.19)$$

(2) Replace each  $L$ -function by its corresponding “approximate” functional equation (6.2.3). Multiply out to get an expression of the form:

$$(\text{product of } \mathcal{X}_D(s) \text{ factors}) \sum_{\substack{n_1, \dots, n_k \\ n_i \text{ monic}}} (\text{summand}). \quad (6.2.20)$$

Since  $s = \frac{1}{2} + \alpha_j$  implies that  $1 - s = \frac{1}{2} - \alpha_j$ , we have,

$$\begin{aligned} & Z\left(\frac{1}{2}; \alpha_1, \dots, \alpha_k\right) \\ &= \left( \mathcal{X}_D\left(\frac{1}{2} + \alpha_1\right)^{-1/2} \sum_{\substack{n_1 \text{ monic} \\ \deg(n_1) \leq g}} \frac{\chi_D(n_1)}{|n_1|^{\frac{1}{2} + \alpha_1}} + \mathcal{X}_D\left(\frac{1}{2} - \alpha_1\right)^{-1/2} \sum_{\substack{n_1 \text{ monic} \\ \deg(n_1) \leq g-1}} \frac{\chi_D(n_1)}{|n_1|^{\frac{1}{2} - \alpha_1}} \right) \\ &\times \dots \times \left( \mathcal{X}_D\left(\frac{1}{2} + \alpha_k\right)^{-1/2} \sum_{\substack{n_k \text{ monic} \\ \deg(n_k) \leq g}} \frac{\chi_D(n_k)}{|n_k|^{\frac{1}{2} + \alpha_k}} + \mathcal{X}_D\left(\frac{1}{2} - \alpha_k\right)^{-1/2} \sum_{\substack{n_k \text{ monic} \\ \deg(n_k) \leq g-1}} \frac{\chi_D(n_k)}{|n_k|^{\frac{1}{2} - \alpha_k}} \right). \end{aligned} \quad (6.2.21)$$

So we can write (6.2.21) as,

$$\begin{aligned} & Z\left(\frac{1}{2}; \alpha_1, \dots, \alpha_k\right) \\ &= \prod_{j=1}^k \left( \mathcal{X}_D\left(\frac{1}{2} + \alpha_j\right)^{-1/2} \sum_{\substack{n_j \text{ monic} \\ \deg(n_j) \leq g}} \frac{\chi_D(n_j)}{|n_j|^{\frac{1}{2} + \alpha_j}} + \mathcal{X}_D\left(\frac{1}{2} - \alpha_j\right)^{-1/2} \sum_{\substack{n_j \text{ monic} \\ \deg(n_j) \leq g-1}} \frac{\chi_D(n_j)}{|n_j|^{\frac{1}{2} - \alpha_j}} \right) \\ &= \sum_{\varepsilon_j = \pm 1} \prod_{j=1}^k \left( \mathcal{X}_D\left(\frac{1}{2} + \varepsilon_j \alpha_j\right)^{-1/2} \sum_{\substack{n_j \text{ monic} \\ \deg(n_j) \leq f(\varepsilon_j)}} \frac{\chi_D(n_j)}{|n_j|^{\frac{1}{2} + \varepsilon_j \alpha_j}} \right), \end{aligned} \quad (6.2.22)$$

where  $f(1) = g$  and  $f(-1) = g - 1$ . We then multiply out and end up with,

$$Z\left(\frac{1}{2}; \alpha_1, \dots, \alpha_k\right) = \sum_{\varepsilon_j = \pm 1} \prod_{j=1}^k \mathcal{X}_D\left(\frac{1}{2} + \varepsilon_j \alpha_j\right)^{-1/2} \sum_{\substack{n_1, \dots, n_k \\ n_i \text{ monic}}} \frac{\chi_D(n_1 \dots n_k)}{\prod_{j=1}^k |n_j|^{\frac{1}{2} + \varepsilon_j \alpha_j}}. \quad (6.2.23)$$

**(3) Replace the product of  $\varepsilon_f$ -factors by its average over the family.**

Note that in this case the  $\varepsilon_f$ -factors are equal to 1 and therefore do not produce any effect on the final result.

**(4) Replace each summand by its expected value when averaged over the family  $\mathcal{H}_{2g+1,q}$ .**

In this step we need to average over all fundamental discriminants  $D \in \mathcal{H}_{2g+1,q}$  and as a preliminary task, we will restate the following orthogonality relation for quadratic Dirichlet characters over function fields.

**Lemma 6.2.3.** *Let*

$$a_m = \prod_{\substack{P \text{ monic} \\ \text{irreducible} \\ P|m}} \left(1 + \frac{1}{|P|}\right)^{-1}. \quad (6.2.24)$$

Then,

$$\lim_{\deg(D) \rightarrow \infty} \frac{1}{\#\mathcal{H}_{2g+1,q}} \sum_{D \in \mathcal{H}_{2g+1,q}} \chi_D(m) = \begin{cases} a_m & \text{if } m = \square \\ 0 & \text{otherwise.} \end{cases} \quad (6.2.25)$$

*Proof.* We start by considering  $m = \square = l^2$ , then using Proposition 3.4.3 and the fact that  $\frac{\Phi(l)}{|l|} < 1$  we have,

$$\sum_{D \in \mathcal{H}_{2g+1,q}} \chi_D(m) = \frac{|D|}{\zeta_A(2)} \prod_{\substack{P \text{ monic} \\ \text{irreducible} \\ P|m}} \left(1 + \frac{1}{|P|}\right)^{-1} + O(\sqrt{|D|}).$$

So,

$$\begin{aligned} \frac{1}{\#\mathcal{H}_{2g+1,q}} \sum_{D \in \mathcal{H}_{2g+1,q}} \chi_D(m = l^2) &= \frac{|D|}{\zeta_A(2)} \frac{1}{(q-1)q^{2g}} \prod_{\substack{P \text{ monic} \\ \text{irreducible} \\ P|m}} \left(1 + \frac{1}{|P|}\right)^{-1} \\ &+ O(\sqrt{|D|}((q-1)q^{2g})^{-1}) \\ &= \prod_{\substack{P \text{ monic} \\ \text{irreducible} \\ P|m}} \left(1 + \frac{1}{|P|}\right)^{-1} + O(q^{-g}). \end{aligned}$$

Therefore,

$$\lim_{\deg(D) \rightarrow \infty} \frac{1}{\#\mathcal{H}_{2g+1,q}} \sum_{D \in \mathcal{H}_{2g+1,q}} \chi_D(m = l^2) = \prod_{\substack{P \text{ monic} \\ \text{irreducible} \\ P|m}} \left(1 + \frac{1}{|P|}\right)^{-1}.$$

If  $m \neq \square$  we can use the function field version of the Pólya–Vinogradov inequality [Fai-Rud, Lemma 2.1] to bound the sum over non-trivial Dirichlet characters,

$$\left| \sum_{\substack{D \in \mathcal{H}_{2g+1,q} \\ m \neq \square}} \chi_D(m) \right| \ll 2^{\deg(m)} \sqrt{|D|},$$

and so we end up with,

$$\begin{aligned} \frac{1}{\#\mathcal{H}_{2g+1,q}} \sum_{\substack{D \in \mathcal{H}_{2g+1,q} \\ m \neq \square}} \chi_D(m) &\ll \frac{2^g \sqrt{|D|}}{(q-1)q^{2g}} \\ &\ll q^{-g} 2^g, \end{aligned}$$

which tends to zero when  $g \rightarrow \infty$  since  $q > 3$  is fixed.  $\square$

Using Lemma 6.2.3, we can average the summand in (6.2.23), since

$$\lim_{g \rightarrow \infty} \langle \chi_D(n_1) \dots \chi_D(n_k) \rangle = \begin{cases} \prod_{P|\square} \left(1 + \frac{1}{|P|}\right)^{-1} & \text{if } n_1 \dots n_k = \square, \\ 0 & \text{otherwise.} \end{cases} \quad (6.2.26)$$

We therefore have

$$\begin{aligned} \lim_{g \rightarrow \infty} \frac{1}{\#\mathcal{H}_{2g+1,q}} \sum_{D \in \mathcal{H}_{2g+1,q}} \sum_{\substack{n_1, \dots, n_k \\ n_i \text{ monic}}} \frac{\chi_D(n_1 \dots n_k)}{\prod_{j=1}^k |n_j|^{\frac{1}{2} + \varepsilon_j \alpha_j}} &= \sum_{\substack{n_1, \dots, n_k \\ n_i \text{ monic} \\ n_1 \dots n_k = m^2}} \frac{a_{m^2}}{\prod_{j=1}^k |n_j|^{\frac{1}{2} + \varepsilon_j \alpha_j}} \\ &= \sum_{m \text{ monic}} \sum_{\substack{n_1, \dots, n_k \\ n_i \text{ monic} \\ n_1 \dots n_k = m^2}} \frac{a_{m^2}}{\prod_{j=1}^k |n_j|^{\frac{1}{2} + \varepsilon_j \alpha_j}}. \end{aligned}$$

**(5) Extend each of  $n_1, \dots, n_k$  sum for all monic polynomials and denote the result  $M(s; \alpha_1, \dots, \alpha_k)$ .**

If we call

$$R_k \left( \frac{1}{2}; \varepsilon_1 \alpha_1, \dots, \varepsilon_k \alpha_k \right) = \sum_{m \text{ monic}} \sum_{\substack{n_1, \dots, n_k \\ n_i \text{ monic} \\ n_1 \dots n_k = m^2}} \frac{a_{m^2}}{\prod_{j=1}^k |n_j|^{\frac{1}{2} + \varepsilon_j \alpha_j}}, \quad (6.2.27)$$

we have that the quantity produced by the recipe is



$$M\left(\frac{1}{2}; \alpha_1 \dots \alpha_k\right) = \sum_{\varepsilon_j = \pm 1} \prod_{j=1}^k \mathcal{X}_D(\frac{1}{2} + \varepsilon_j \alpha_j)^{-1/2} R_k\left(\frac{1}{2}; \varepsilon_1 \alpha_1, \dots, \varepsilon_k \alpha_k\right). \quad (6.2.28)$$

(6) The conjecture is

$$\sum_{D \in \mathcal{H}_{2g+1, q}} Z(\frac{1}{2}; \alpha_1, \dots, \alpha_k) = \sum_{D \in \mathcal{H}_{2g+1, q}} M(\frac{1}{2}; \alpha_1, \dots, \alpha_k) (1 + O(|D|^{-\frac{1}{2} + \varepsilon})), \quad (6.2.29)$$

for all  $\varepsilon > 0$ .

## 6.3 Putting the Conjecture in a More Useful Form

The conjecture (6.2.29) is problematic in the form presented because the individual terms have poles that cancel when summed. In this section we put it in a more useful form, writing  $R_k$  as an Euler product and then factoring out the appropriate  $\zeta_A(s)$ -factors.

We have that  $a_m$  is multiplicative, since

$$a_{mn} = a_m a_n, \quad \text{where } a_m = \prod_{\substack{P \text{ monic} \\ \text{irreducible} \\ P|m}} (1 + |P|^{-1})^{-1},$$

and if we define

$$\psi(x) := \sum_{\substack{n_1 \dots n_k = x \\ n_i \text{ monic}}} \frac{1}{|n_1|^{s+\alpha_1} \dots |n_k|^{s+\alpha_k}},$$

we have that  $\psi(m^2)$  is multiplicative on  $m$ .

So,

$$\begin{aligned}
 & \sum_{m \text{ monic}} \sum_{\substack{n_1, \dots, n_k \\ n_i \text{ monic} \\ n_1 \dots n_k = m^2}} \frac{a_{m^2}}{|n_1|^{s+\alpha_1} \dots |n_k|^{s+\alpha_k}} \\
 &= \sum_{m \text{ monic}} a_{m^2} \sum_{\substack{n_1, \dots, n_k \\ n_i \text{ monic} \\ n_1 \dots n_k = m^2}} \frac{1}{|n_1|^{s+\alpha_1} \dots |n_k|^{s+\alpha_k}} \\
 &= \sum_{m \text{ monic}} a_{m^2} \psi(m^2) = \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left( 1 + \sum_{j=1}^{\infty} a_{P^{2j}} \psi(P^{2j}) \right),
 \end{aligned}$$

where

$$\psi(P^{2j}) = \sum_{\substack{n_1, \dots, n_k \\ n_i \text{ monic} \\ n_1 \dots n_k = P^{2j}}} \frac{1}{|n_1|^{s+\alpha_1} \dots |n_k|^{s+\alpha_k}},$$

and so,  $n_i = P^{e_i}$ , for  $i = 1, \dots, k$  and  $e_1 + \dots + e_k = 2j$  due the fact that  $|P|^{e_1} \dots |P|^{e_k} = |P|^{e_1 + \dots + e_k} = |P|^{2j}$ .

Hence we can write

$$\begin{aligned}
 \psi(P^{2j}) &= \sum_{\substack{n_1, \dots, n_k \\ n_i \text{ monic} \\ n_1 \dots n_k = P^{2j}}} \frac{1}{|P|^{e_1(s+\alpha_1)} \dots |P|^{e_k(s+\alpha_k)}} \\
 &= \sum_{\substack{e_1, \dots, e_k \geq 0 \\ e_1 + \dots + e_k = 2j}} \frac{1}{|P|^{e_1(s+\alpha_1)} \dots |P|^{e_k(s+\alpha_k)}} \\
 &= \sum_{\substack{e_1, \dots, e_k \geq 0 \\ e_1 + \dots + e_k = 2j}} \prod_{i=1}^k \frac{1}{|P|^{e_i(s+\alpha_i)}}
 \end{aligned}$$

and thus we end up with

$$\begin{aligned}
 R_k(s; \alpha_1, \dots, \alpha_k) &= \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left( 1 + \sum_{j=1}^{\infty} a_{P^{2j}} \psi(P^{2j}) \right) \\
 &= \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left( 1 + \sum_{j=1}^{\infty} a_{P^{2j}} \sum_{\substack{e_1, \dots, e_k \geq 0 \\ e_1 + \dots + e_k = 2j}} \prod_{i=1}^k \frac{1}{|P|^{e_i(s+\alpha_i)}} \right), \quad (6.3.1)
 \end{aligned}$$

and as

$$a_{P^{2j}} = (1 + |P|^{-1})^{-1},$$

we have that (6.3.1) becomes

$$\begin{aligned}
 R_k(s; \alpha_1, \dots, \alpha_k) &= \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left( 1 + (1 + |P|^{-1})^{-1} \sum_{j=1}^{\infty} \sum_{\substack{e_1, \dots, e_k \geq 0 \\ e_1 + \dots + e_k = 2j}} \prod_{i=1}^k \frac{1}{|P|^{e_i(s+\alpha_i)}} \right) \\
 &= \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} R_{k,P}. \tag{6.3.2}
 \end{aligned}$$

Using

$$(1 + |P|^{-1})^{-1} = 1 - \frac{1}{|P|} + \frac{1}{|P|^2} - \frac{1}{|P|^3} + \dots = \sum_{l=0}^{\infty} \frac{(-1)^l}{|P|^l}$$

we have that

$$R_{k,P} = 1 + \sum_{l=0}^{\infty} \sum_{j=1}^{\infty} \sum_{\substack{e_1, \dots, e_k \geq 0 \\ e_1 + \dots + e_k = 2j}} \prod_{i=1}^k \frac{(-1)^l}{|P|^{e_i(s+\alpha_i)+l}}$$

and so

$$R(s; \alpha_1, \dots, \alpha_k) = \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left( 1 + \sum_{l=0}^{\infty} \sum_{j=1}^{\infty} \sum_{\substack{e_1, \dots, e_k \geq 0 \\ e_1 + \dots + e_k = 2j}} \prod_{i=1}^k \frac{(-1)^l}{|P|^{e_i(s+\alpha_i)+l}} \right).$$

The key point is that only the terms with  $e_1 + \dots + e_k = 2$  produce poles.

Thus, we look for  $l = 0$  and  $j = 1$  and have

$$\begin{aligned}
 R_{k,P} &= 1 + \sum_{e_1 + \dots + e_k = 2} \prod_{i=1}^k \frac{1}{|P|^{e_i(s+\alpha_i)}} + (\text{lower order terms}) \\
 &= 1 + \sum_{e_1 + \dots + e_k = 2} \left( \frac{1}{|P|^{e_1(s+\alpha_1)}} \cdots \frac{1}{|P|^{e_k(s+\alpha_k)}} \right) + (\text{lower order terms}) \\
 &= 1 + \frac{1}{|P|^{(s+\alpha_1)}|P|^{s+\alpha_2}} + \frac{1}{|P|^{s+\alpha_1}|P|^{s+\alpha_3}} + \dots + (\text{lower order terms}) \\
 &= 1 + \sum_{1 \leq i \leq j \leq k} \frac{1}{|P|^{2s+\alpha_i+\alpha_j}} + (\text{lower order terms})
 \end{aligned}$$

Hence we can write

$$R_{k,P} = 1 + \sum_{1 \leq i \leq j \leq k} \frac{1}{|P|^{2s+\alpha_i+\alpha_j}} + O(|P|^{-1-2s+\varepsilon}) + O(|P|^{-3s+\varepsilon})$$

(for more details see [CFKRS05, pg87]). Expressing  $R_{k,P}$  as a product, we finish with

$$R_{k,P} = \prod_{1 \leq i \leq j \leq k} \left( 1 + \frac{1}{|P|^{2s+\alpha_i+\alpha_j}} \right) \times (1 + O(|P|^{-1-2s+\varepsilon}) + O(|P|^{-3s+\varepsilon})).$$

Now, since

$$\prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left( 1 + \frac{1}{|P|^{2s}} \right) = \frac{\zeta_A(2s)}{\zeta_A(4s)}$$

has a simple pole at  $s = \frac{1}{2}$  and

$$\prod_{\substack{P \text{ monic} \\ \text{irreducible}}} (1 + O(|P|^{-1-2s+\varepsilon}) + O(|P|^{-3s+\varepsilon}))$$

is analytic in  $\Re(s) > \frac{1}{3}$ , we see that  $\prod_P R_{k,P}$  has a pole at  $s = \frac{1}{2}$  of order  $k(k+1)/2$  if  $\alpha_1 = \dots = \alpha_k = 0$ .

With the divergent sums replaced by their analytic continuation and the leading order poles clearly identified, we are almost ready to put the Conjecture 6.2.29 in a more desirable form. We just need to factor out the appropriate zeta-factors and write the above product  $\prod_P R_{k,P}$  as

$$\begin{aligned} & R_k(s; \alpha_1, \dots, \alpha_k) \\ &= \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left( \prod_{1 \leq i \leq j \leq k} \left( 1 + \frac{1}{|P|^{2s+\alpha_i+\alpha_j}} \right) (1 + O(|P|^{-1-2s+\varepsilon}) + O(|P|^{-3s+\varepsilon})) \right) \\ &= \prod_{1 \leq i \leq j \leq k} \frac{\zeta_A(2s + \alpha_i + \alpha_j)}{\zeta_A(2(2s + \alpha_i + \alpha_j))} \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} (1 + O(|P|^{-1-2s+\varepsilon}) + O(|P|^{-3s+\varepsilon})) \\ &= \prod_{1 \leq i \leq j \leq k} \zeta_A(2s + \alpha_i + \alpha_j) \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \prod_{1 \leq i \leq j \leq k} \left( 1 - \frac{1}{|P|^{2s+\alpha_i+\alpha_j}} \right) \\ &\quad \times \left( 1 + \frac{1}{|P|^{2s+\alpha_i+\alpha_j}} \right) (1 + O(|P|^{-1-2s+\varepsilon}) + O(|P|^{-3s+\varepsilon})) \\ &= \prod_{1 \leq i \leq j \leq k} \zeta_A(2s + \alpha_i + \alpha_j) \\ &\quad \times \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left( R_{k,P}(s; \alpha_1, \dots, \alpha_k) \prod_{1 \leq i \leq j \leq k} \left( 1 - \frac{1}{|P|^{2s+\alpha_i+\alpha_j}} \right) \right) \\ &= \prod_{1 \leq i \leq j \leq k} \zeta_A(2s + \alpha_i + \alpha_j) A_k(s; \alpha_1, \dots, \alpha_k). \end{aligned}$$

Here,  $A_k$  defines an absolutely convergent Dirichlet series for  $\Re(s) > \frac{1}{2} + \delta$  for some  $\delta > 0$  and for all  $\alpha_j$ 's in some sufficiently small neighborhood of 0. Consequently, we have

$$\begin{aligned} & M\left(\frac{1}{2}; \alpha_1, \dots, \alpha_k\right) \\ &= \sum_{\varepsilon_j = \pm 1} \prod_{j=1}^k \mathcal{X}_D\left(\frac{1}{2} + \varepsilon_j \alpha_j\right)^{-\frac{1}{2}} \prod_{1 \leq i \leq j \leq k} \zeta_A(1 + \varepsilon_i \alpha_i + \varepsilon_j \alpha_j) A_k\left(\frac{1}{2}; \varepsilon_1 \alpha_1, \dots, \varepsilon_k \alpha_k\right) \end{aligned} \quad (6.3.3)$$

and so the conjectured asymptotic takes the form

$$\begin{aligned} & \sum_{D \in \mathcal{H}_{2g+1, q}} Z\left(\frac{1}{2}, \alpha_1, \dots, \alpha_k\right) \\ &= \sum_{D \in \mathcal{H}_{2g+1, q}} \sum_{\varepsilon_j = \pm 1} \prod_{j=1}^k \mathcal{X}_D\left(\frac{1}{2} + \varepsilon_j \alpha_j\right)^{-\frac{1}{2}} A_k\left(\frac{1}{2}; \varepsilon_1 \alpha_1, \dots, \varepsilon_k \alpha_k\right) \\ & \quad \times \prod_{1 \leq i \leq j \leq k} \zeta_A(1 + \varepsilon_i \alpha_i + \varepsilon_j \alpha_j) (1 + O(|D|^{-\frac{1}{2} + \varepsilon})) \end{aligned} \quad (6.3.4)$$

Using the definition of  $\mathcal{X}_D(s)$ , we have that

$$\mathcal{X}_D\left(\frac{1}{2} + \varepsilon_j \alpha_j\right)^{-\frac{1}{2}} = |D|^{\frac{\varepsilon_j \alpha_j}{2}} \mathcal{X}\left(\frac{1}{2} + \varepsilon_j \alpha_j\right)^{-\frac{1}{2}}, \quad (6.3.5)$$

and substituting this into (6.3.4), after some arithmetical manipulations we are led to the following form of the conjecture:

$$\begin{aligned} & \sum_{D \in \mathcal{H}_{2g+1, q}} Z_{\mathcal{L}}\left(\frac{1}{2} + \alpha_1, \chi_D\right) \dots Z_{\mathcal{L}}\left(\frac{1}{2} + \alpha_k, \chi_D\right) \\ &= \sum_{\varepsilon_j = \pm 1} \prod_{j=1}^k \mathcal{X}\left(\frac{1}{2} + \varepsilon_j \alpha_j\right)^{-1/2} \\ & \quad \times \sum_{D \in \mathcal{H}_{2g+1, q}} R_k\left(\frac{1}{2}; \varepsilon_1 \alpha_1, \dots, \varepsilon_k \alpha_k\right) |D|^{\left(\frac{1}{2}\right) \sum_{j=1}^k \varepsilon_j \alpha_j} (1 + O(|D|^{-\frac{1}{2} + \varepsilon})). \end{aligned} \quad (6.3.6)$$

Note that (6.3.6) can be seen as the function field analogue of the formula (4.4.22) in [CFKRS05].

**Lemma 6.3.1.** *We have that,*

$$\begin{aligned}
 A_k\left(\frac{1}{2}; \alpha_1, \dots, \alpha_k\right) &= \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \prod_{1 \leq i < j \leq k} \left(1 - \frac{1}{|P|^{1+z_i+z_j}}\right) \\
 &\times \left(\frac{1}{2} \left(\prod_{j=1}^k \left(1 - \frac{1}{|P|^{\frac{1}{2}+z_j}}\right)^{-1} + \prod_{j=1}^k \left(1 + \frac{1}{|P|^{\frac{1}{2}+z_j}}\right)^{-1}\right) + \frac{1}{|P|}\right) \\
 &\times \left(1 + \frac{1}{|P|}\right)^{-1}. \tag{6.3.7}
 \end{aligned}$$

*Proof.* We can write,

$$\begin{aligned}
 &A_k\left(\frac{1}{2}; \alpha_1, \dots, \alpha_k\right) \\
 &= \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left(R_{k,P}\left(\frac{1}{2}; \alpha_1, \dots, \alpha_k\right) \prod_{1 \leq i < j \leq k} \left(1 - \frac{1}{|P|^{1+\alpha_i+\alpha_j}}\right)\right) \\
 &= \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \prod_{1 \leq i < j \leq k} \left(1 - \frac{1}{|P|^{1+\alpha_i+\alpha_j}}\right) R_{k,P}\left(\frac{1}{2}; \alpha_1, \dots, \alpha_k\right) \\
 &= \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \prod_{1 \leq i < j \leq k} \left(1 - \frac{1}{|P|^{1+\alpha_i+\alpha_j}}\right) \\
 &\times \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left(1 + (1 + |P|^{-1})^{-1} \sum_{j=1}^{\infty} \sum_{\substack{e_1, \dots, e_k \geq 0 \\ e_1 + \dots + e_k = 2j}} \prod_{i=1}^k \frac{1}{|P|^{e_i(\frac{1}{2}+\alpha_i)}}\right). \tag{6.3.8}
 \end{aligned}$$

Making the following substitution

$$\begin{aligned}
 &1 + (1 + |P|^{-1})^{-1} \sum_{j=1}^{\infty} \sum_{\substack{e_1, \dots, e_k \geq 0 \\ e_1 + \dots + e_k = 2j}} \prod_{i=1}^k \frac{1}{|P|^{e_i(\frac{1}{2}+\alpha_i)}} \\
 &= (1 + |P|^{-1})^{-1} \sum_{j=0}^{\infty} \sum_{\substack{e_1, \dots, e_k \geq 0 \\ e_1 + \dots + e_k = 2j}} \prod_{i=1}^k \frac{1}{|P|^{e_i(\frac{1}{2}+\alpha_i)}} \tag{6.3.9}
 \end{aligned}$$

and writing,

$$\begin{aligned}
 & \sum_{j=0}^{\infty} \sum_{\substack{e_1, \dots, e_k \geq 0 \\ e_1 + \dots + e_k = 2j}} \prod_{i=1}^k \frac{1}{|P|^{e_i(\frac{1}{2} + \alpha_i)}} \\
 &= \sum_{j=0}^{\infty} \frac{1}{2} \left( \sum_{\substack{e_1, \dots, e_k \geq 0 \\ e_1 + \dots + e_k = 2j}} \prod_{i=1}^k \left( \frac{1}{|P|^{\frac{1}{2} + \alpha_i}} \right)^{e_i} + \sum_{\substack{e_1, \dots, e_k \geq 0 \\ e_1 + \dots + e_k = 2j}} \prod_{i=1}^k \left( \frac{1}{|P|^{\frac{1}{2} + \alpha_i}} \right)^{e_i} \right) \\
 &= \frac{1}{2} \left( \sum_{e_1, \dots, e_k \geq 0} \prod_{i=1}^k \left( \frac{1}{|P|^{\frac{1}{2} + \alpha_i}} \right)^{e_i} + \sum_{e_1, \dots, e_k \geq 0} (-1)^{e_1 + \dots + e_k} \prod_{i=1}^k \left( \frac{1}{|P|^{\frac{1}{2} + \alpha_i}} \right)^{e_i} \right) \\
 &= \frac{1}{2} \left( \sum_{e_1, \dots, e_k \geq 0} \prod_{i=1}^k \left( \frac{1}{|P|^{\frac{1}{2} + \alpha_i}} \right)^{e_i} + \sum_{e_1, \dots, e_k \geq 0} \prod_{i=1}^k (-1)^{e_i} \left( \frac{1}{|P|^{\frac{1}{2} + \alpha_i}} \right)^{e_i} \right) \\
 &= \frac{1}{2} \left( \prod_{i=1}^k \sum_{e_i=0}^{\infty} \left( \frac{1}{|P|^{\frac{1}{2} + \alpha_i}} \right)^{e_i} + \prod_{i=1}^k \sum_{e_i=0}^{\infty} (-1)^{e_i} \left( \frac{1}{|P|^{\frac{1}{2} + \alpha_i}} \right)^{e_i} \right) \\
 &= \frac{1}{2} \left( \prod_{i=1}^k \left( 1 - \frac{1}{|P|^{\frac{1}{2} + \alpha_j}} \right)^{-1} + \prod_{i=1}^k \left( 1 + \frac{1}{|P|^{\frac{1}{2} + \alpha_j}} \right) \right)^{-1}, \tag{6.3.10}
 \end{aligned}$$

we obtain the formula stated in the lemma.  $\square$

## 6.4 The Final Form of the Conjecture

In this section we will use the following lemma from [CFKRS05].

**Lemma 6.4.1** (Conrey, Farmer, Keating, Rubinstein and Snaith). *Suppose  $F$  is a symmetric function of  $k$  variables, regular near  $(0, \dots, 0)$ , and that  $f(s)$  has a simple pole of residue 1 at  $s = 0$  and is otherwise analytic in a neighborhood of  $s = 0$ , and let*

$$K(a_1, \dots, a_k) = F(a_1, \dots, a_k) \prod_{1 \leq i \leq j \leq k} f(a_i + a_j) \tag{6.4.1}$$

or

$$K(a_1, \dots, a_k) = F(a_1, \dots, a_k) \prod_{1 \leq i < j \leq k} f(a_i + a_j). \tag{6.4.2}$$

If  $\alpha_i + \alpha_j$  are contained in the region of analyticity of  $f(s)$  then

$$\begin{aligned} \sum_{\epsilon_j = \pm 1} K(\epsilon_1 \alpha_1, \dots, \epsilon_k \alpha_k) &= \frac{(-1)^{k(k-1)/2} 2^k}{(2\pi i)^k k!} \oint \cdots \oint K(z_1, \dots, z_k) \\ &\times \frac{\Delta(z_1^2, \dots, z_k^2)^2 \prod_{j=1}^k z_j}{\prod_{i=1}^k \prod_{j=1}^k (z_i - \alpha_j)(z_i + \alpha_j)} dz_1 \cdots dz_k, \end{aligned}$$

and

$$\begin{aligned} \sum_{\epsilon_j = \pm 1} \left( \prod_{j=1}^k \epsilon_j \right) K(\epsilon_1 \alpha_1, \dots, \epsilon_k \alpha_k) \\ &= \frac{(-1)^{k(k-1)/2} 2^k}{(2\pi i)^k k!} \oint \cdots \oint K(z_1, \dots, z_k) \\ &\times \frac{\Delta(z_1^2, \dots, z_k^2)^2 \prod_{j=1}^k \alpha_j}{\prod_{i=1}^k \prod_{j=1}^k (z_i - \alpha_j)(z_i + \alpha_j)} dz_1 \cdots dz_k, \end{aligned}$$

where the path of integration encloses the  $\pm \alpha_j$ 's.

We will use this lemma to write the conjecture for function fields as a contour integral. For this, note that

$$\begin{aligned} \sum_{D \in \mathcal{H}_{2g+1, q}} Z_{\mathcal{L}}(\tfrac{1}{2} + \alpha_1, \chi_D) \cdots Z_{\mathcal{L}}(\tfrac{1}{2} + \alpha_k, \chi_D) \\ &= \sum_{D \in \mathcal{H}_{2g+1, q}} \prod_{j=1}^k \mathcal{X}_D(\tfrac{1}{2} + \alpha_j)^{-1/2} L(\tfrac{1}{2} + \alpha_1, \chi_D) \cdots L(\tfrac{1}{2} + \alpha_k, \chi_D) \quad (6.4.3) \end{aligned}$$

and as  $\mathcal{X}_D(\frac{1}{2} + \alpha_j)^{-1/2}$  does not depend on  $D$ , we can factor it out and write the following expression:



$$\begin{aligned}
 & \sum_{D \in \mathcal{H}_{2g+1,q}} L\left(\frac{1}{2} + \alpha_1, \chi_D\right) \dots L\left(\frac{1}{2} + \alpha_k, \chi_D\right) \\
 = & \sum_{D \in \mathcal{H}_{2g+1,q}} \sum_{\varepsilon_j = \pm 1} \prod_{j=1}^k \mathcal{X}\left(\frac{1}{2} + \varepsilon_j \alpha_j\right)^{-1/2} \prod_{j=1}^k \mathcal{X}_D\left(\frac{1}{2} + \varepsilon_j \alpha_j\right)^{1/2} R_k\left(\frac{1}{2}; \varepsilon_1 \alpha_1, \dots, \varepsilon_k \alpha_k\right) \\
 & \quad \times |D|^{\frac{1}{2} \sum_{j=1}^k \varepsilon_j \alpha_j} (1 + O(|D|^{-\frac{1}{2} + \varepsilon})) \\
 = & \sum_{D \in \mathcal{H}_{2g+1,q}} \prod_{j=1}^k \mathcal{X}\left(\frac{1}{2} + \alpha_j\right) |D|^{-\frac{1}{2} \sum_{j=1}^k \alpha_j} \sum_{\varepsilon_j = \pm 1} \prod_{j=1}^k \mathcal{X}\left(\frac{1}{2} + \varepsilon_j \alpha_j\right)^{-1/2} \\
 & \quad \times R_k\left(\frac{1}{2}; \varepsilon_1 \alpha_1, \dots, \varepsilon_k \alpha_k\right) |D|^{\frac{1}{2} \sum_{j=1}^k \varepsilon_j \alpha_j} (1 + O(|D|^{-\frac{1}{2} + \varepsilon})) \\
 = & \sum_{D \in \mathcal{H}_{2g+1,q}} \prod_{j=1}^k \mathcal{X}\left(\frac{1}{2} + \alpha_j\right) |D|^{-\frac{1}{2} \sum_{j=1}^k \alpha_j} \sum_{\varepsilon_j = \pm 1} \prod_{j=1}^k \mathcal{X}\left(\frac{1}{2} + \varepsilon_j \alpha_j\right)^{-1/2} \\
 & \times A_k\left(\frac{1}{2}; \alpha_1, \dots, \alpha_k\right) |D|^{\frac{1}{2} \sum_{j=1}^k \varepsilon_j \alpha_j} \prod_{1 \leq i \leq j \leq k} \zeta_A(1 + \varepsilon_i \alpha_i + \varepsilon_j \alpha_j) (1 + O(|D|^{-\frac{1}{2} + \varepsilon}))
 \end{aligned} \tag{6.4.4}$$

Multiplying and dividing by  $(\log q)^{k(k+1)/2}$  we have that

$$\begin{aligned}
 & \sum_{D \in \mathcal{H}_{2g+1,q}} L\left(\frac{1}{2} + \alpha_1, \chi_D\right) \dots L\left(\frac{1}{2} + \alpha_k, \chi_D\right) \\
 = & \sum_{D \in \mathcal{H}_{2g+1,q}} \frac{\prod_{j=1}^k \mathcal{X}\left(\frac{1}{2} + \alpha_j\right) |D|^{-\frac{1}{2} \sum_{j=1}^k \alpha_j}}{(\log q)^{k(k+1)/2}} \sum_{\varepsilon_j = \pm 1} \prod_{j=1}^k \mathcal{X}\left(\frac{1}{2} + \varepsilon_j \alpha_j\right)^{-1/2} \\
 & \times A_k\left(\frac{1}{2}; \alpha_1, \dots, \alpha_k\right) |D|^{\frac{1}{2} \sum_{j=1}^k \varepsilon_j \alpha_j} \prod_{1 \leq i \leq j \leq k} \zeta_A(1 + \varepsilon_i \alpha_i + \varepsilon_j \alpha_j) (\log q) \\
 & \quad \times (1 + O(|D|^{-\frac{1}{2} + \varepsilon})). \tag{6.4.5}
 \end{aligned}$$

If we call

$$F(\alpha_1, \dots, \alpha_k) = \prod_{j=1}^k \mathcal{X}\left(\frac{1}{2} + \alpha_j\right)^{-1/2} A_k\left(\frac{1}{2}; \alpha_1, \dots, \alpha_k\right) |D|^{\frac{1}{2} \sum_{j=1}^k \alpha_j}, \tag{6.4.6}$$

and

$$f(s) = \zeta_A(1 + s) \log q \quad \text{and so} \quad f(\alpha_i + \alpha_j) = \zeta_A(1 + \alpha_i + \alpha_j) \log q, \tag{6.4.7}$$

we have that  $f(s)$  has a simple pole at  $s = 0$  with residue 1.

Denoting

$$K(\alpha_1, \dots, \alpha_k) = F(\alpha_1, \dots, \alpha_k) \prod_{1 \leq i \leq j \leq k} f(\alpha_i + \alpha_j), \quad (6.4.8)$$

we can write (6.4.5) as

$$\left( \sum_{D \in \mathcal{H}_{2g+1,q}} \frac{\prod_{j=1}^k \mathcal{X}(\frac{1}{2} + \alpha_j) |D|^{-\frac{1}{2} \sum_{j=1}^k \alpha_j}}{(\log q)^{k(k+1)/2}} \sum_{\varepsilon_j = \pm 1} K(\varepsilon_1 \alpha_1, \dots, \varepsilon_k \alpha_k) \right) \times (1 + O(|D|^{-\frac{1}{2} + \varepsilon})), \quad (6.4.9)$$

and now we can use Lemma 6.4.1 and write (6.4.9) as,

$$\begin{aligned} & \sum_{D \in \mathcal{H}_{2g+1,q}} \frac{\prod_{j=1}^k \mathcal{X}(\frac{1}{2} + \alpha_j) |D|^{-\frac{1}{2} \sum_{j=1}^k \alpha_j}}{(\log q)^{k(k+1)/2}} \frac{(-1)^{k(k-1)/2}}{(2\pi i)^k} \frac{2^k}{k!} \oint \cdots \oint K(z_1, \dots, z_k) \\ & \quad \times \frac{\Delta(z_1^2, \dots, z_k^2)^2 \prod_{j=1}^k z_j}{\prod_{i=1}^k \prod_{j=1}^k (z_i - \alpha_j)(z_i + \alpha_j)} dz_1 \cdots dz_k (1 + O(|D|^{-\frac{1}{2} + \varepsilon})) \\ & = \sum_{D \in \mathcal{H}_{2g+1,q}} \prod_{j=1}^k \mathcal{X}(\frac{1}{2} + \alpha_j) |D|^{-\frac{1}{2} \sum_{j=1}^k \alpha_j} \frac{(-1)^{k(k-1)/2}}{(2\pi i)^k} \frac{2^k}{k!} \oint \cdots \oint F(z_1, \dots, z_k) \\ & \times \prod_{1 \leq i \leq j \leq k} \zeta_A(1 + z_i + z_j) \frac{\Delta(z_1^2, \dots, z_k^2)^2 \prod_{j=1}^k z_j}{\prod_{i=1}^k \prod_{j=1}^k (z_i - \alpha_j)(z_i + \alpha_j)} dz_1 \cdots dz_k + O(|D|^{\frac{1}{2} + \varepsilon}). \end{aligned} \quad (6.4.10)$$

If we denote now

$$K(z_1, \dots, z_k) = F(z_1, \dots, z_k) \prod_{1 \leq i \leq j \leq k} \zeta_A(1 + z_i + z_j), \quad (6.4.11)$$

we have that (6.4.10) becomes

$$\begin{aligned} & \sum_{D \in \mathcal{H}_{2g+1,q}} \prod_{j=1}^k \mathcal{X}(\frac{1}{2} + \alpha_j) |D|^{-\frac{1}{2} \sum_{j=1}^k \alpha_j} \frac{(-1)^{k(k-1)/2}}{(2\pi i)^k} \frac{2^k}{k!} \oint \cdots \oint K(z_1, \dots, z_k) \\ & \quad \times \frac{\Delta(z_1^2, \dots, z_k^2)^2 \prod_{j=1}^k z_j}{\prod_{i=1}^k \prod_{j=1}^k (z_i - \alpha_j)(z_i + \alpha_j)} dz_1 \cdots dz_k + O(|D|^{\frac{1}{2} + \varepsilon}), \end{aligned} \quad (6.4.12)$$

and if we denote

$$G(z_1, \dots, z_k) = \prod_{j=1}^k \mathcal{X}(\frac{1}{2} + z_j)^{-1/2} A_k(\frac{1}{2}; z_1, \dots, z_k) \prod_{1 \leq i \leq j \leq k} \zeta_A(1 + z_i + z_j), \quad (6.4.13)$$

we have that (6.4.12) is

$$\begin{aligned} & \sum_{D \in \mathcal{H}_{2g+1, q}} \prod_{j=1}^k \mathcal{X}(\frac{1}{2} + \alpha_j) |D|^{-\frac{1}{2} \sum_{j=1}^k \alpha_j} \frac{(-1)^{k(k-1)/2} 2^k}{(2\pi i)^k k!} \oint \cdots \oint G(z_1, \dots, z_k) \\ & \times |D|^{\frac{1}{2} \sum_{j=1}^k z_j} \frac{\Delta(z_1^2, \dots, z_k^2)^2 \prod_{j=1}^k z_j}{\prod_{i=1}^k \prod_{j=1}^k (z_i - \alpha_j)(z_i + \alpha_j)} dz_1 \cdots dz_k + O(|D|^{\frac{1}{2} + \varepsilon}). \end{aligned} \quad (6.4.14)$$

Now calling

$$\begin{aligned} Q_k(x) &= \frac{(-1)^{k(k-1)/2} 2^k}{(2\pi i)^k k!} \oint \cdots \oint G(z_1, \dots, z_k) \\ & \times \frac{\Delta(z_1^2, \dots, z_k^2)^2 \prod_{j=1}^k z_j}{\prod_{i=1}^k \prod_{j=1}^k (z_i - \alpha_j)(z_i + \alpha_j)} q^{\frac{x}{2} \sum_{j=1}^k z_j} dz_1 \cdots dz_k, \end{aligned} \quad (6.4.15)$$

we have established the formulae given in the Conjecture 6.1.3.

**Remark 6.4.2.** *Note that the formulas (6.1.14) and (6.1.18) are the function field analogues of the formulas (1.5.11) and (1.5.15) presented in [CFKRS05] respectively.*

## 6.5 Some Conjectural Values for Moments of $L$ -functions in the Hyperelliptic Ensemble

In this section we use Conjecture 6.1.3 to obtain explicit conjectural values for the first few moments of quadratic Dirichlet  $L$ -functions over function fields.

### 6.5.1 First Moment

We will use Conjecture 6.1.3 to compute the value of the first moment ( $k = 1$ ) of our family of  $L$ -function and compare with the main theorem of Chapter

3. Specifically, we will specialize the formula in Conjecture 6.1.3 for  $k = 1$  to compute

$$\sum_{D \in \mathcal{H}_{2g+1, q}} L\left(\frac{1}{2}, \chi_D\right) = \sum_{D \in \mathcal{H}_{2g+1, q}} Q_1(\log_q |D|)(1 + O(|D|^{-\frac{1}{2}+\epsilon})), \quad (6.5.1)$$

where  $Q_1(x)$  is a polynomial of degree 1, i.e.,  $Q_1(x) = ax + b$ . This will be done using the contour integral formula for  $Q_k(x)$ . We have,

$$Q_1(x) = \frac{1}{\pi i} \oint \frac{G(z_1) \Delta(z_1^2)^2}{z_1} q^{\frac{x}{2} z_1} dz_1 \quad (6.5.2)$$

where

$$G(z_1) = A_k\left(\frac{1}{2}; z_1\right) \mathcal{X}\left(\frac{1}{2} + z_1\right)^{-1/2} \zeta_A(1 + 2z_1). \quad (6.5.3)$$

Remembering that,

$$\Delta(z_1, \dots, z_k) = \prod_{1 \leq i < j \leq k} (z_j - z_i) \quad (6.5.4)$$

is the Vandermonde determinant we have that,

$$\Delta(z_1^2)^2 = 1 \quad (6.5.5)$$

and

$$\mathcal{X}\left(\frac{1}{2} + z_1\right)^{-1/2} = q^{-z_1/2}. \quad (6.5.6)$$

So (6.5.2) becomes,

$$\begin{aligned} \frac{1}{\pi i} \oint \frac{A_k\left(\frac{1}{2}; z_1\right) \mathcal{X}\left(\frac{1}{2} + z_1\right)^{-1/2} \zeta_A(1 + 2z_1)}{z_1} q^{\frac{x}{2} z_1} dz_1 \\ = \frac{1}{\pi i} \oint \frac{A_k\left(\frac{1}{2}; z_1\right) \zeta_A(1 + 2z_1) q^{-z_1/2}}{z_1} q^{\frac{x}{2} z_1} dz_1. \end{aligned} \quad (6.5.7)$$

We also have that,

$$\begin{aligned} A_k\left(\frac{1}{2}; z_1\right) &= \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left(1 - \frac{1}{|P|^{1+2z_1}}\right) \\ &\times \left(\frac{1}{2} \left( \left(1 - \frac{1}{|P|^{1/2+z_1}}\right)^{-1} + \left(1 + \frac{1}{|P|^{1/2+z_1}}\right)^{-1} \right) + \frac{1}{|P|}\right) \\ &\times \left(1 + \frac{1}{|P|}\right)^{-1}. \end{aligned} \quad (6.5.8)$$

Our goal is to compute the integral (6.5.7) where the contour is a small circle around the origin, and for that we need to locate the poles of the integrand,

$$f(z_1) = \frac{A_k(\frac{1}{2}; z_1) \zeta_A(1 + 2z_1) q^{-z_1/2}}{z_1} q^{\frac{x}{2} z_1}. \quad (6.5.9)$$

We note that  $f(z_1)$  has a pole of order 2 at  $z_1 = 0$ . To compute the residue we expand  $f(z_1)$  as a Laurent series and pick up the coefficient of  $1/z_1$ . Expanding the numerator of  $f(z_1)$  around  $z_1 = 0$  we have,

1.

$$A_k(z_1) = A_k(0) + A'_k(0)z_1 + \frac{1}{2}A''_k(0)z_1^2 + \dots$$

2.

$$q^{-z_1/2} = 1 - \frac{1}{2}(\log q)z_1 + \frac{1}{8}(\log q)^2 z_1^2 + \dots$$

3.

$$q^{\frac{x}{2} z_1} = 1 + \frac{1}{2}(\log q)xz_1 + \frac{1}{8}(\log^2 q)x^2 z_1^2 + \dots$$

4.

$$\zeta_A(1 + 2z_1) = \frac{1}{2 \log q} \frac{1}{z_1} + \frac{1}{2} + \frac{1}{6}(\log q)z_1 - \frac{1}{90}(\log^3 q)z_1^3 + \dots$$

Hence we can write,

$$\begin{aligned} f(z_1) &= \left( \frac{A_k(0)}{z_1} + A'_k(0) + \frac{A''_k(0)}{2} z_1 + \dots \right) \\ &\times \left( 1 - \frac{1}{2}(\log q)z_1 + \frac{1}{8}(\log q)^2 z_1^2 + \dots \right) \\ &\times \left( 1 + \frac{1}{2}(\log q)xz_1 + \frac{1}{8}(\log^2 q)x^2 z_1^2 + \dots \right) \\ &\times \left( \frac{1}{2 \log q} \frac{1}{z_1} + \frac{1}{2} + \frac{1}{6}(\log q)z_1 - \frac{1}{90}(\log^3 q)z_1^3 + \dots \right), \end{aligned} \quad (6.5.10)$$

where we have denoted  $A_k(\frac{1}{2}; z_1)$  by  $A_k(z_1)$ .

Multiplying the above expression we identify the coefficient of  $1/z_1$ . Therefore

$$\operatorname{Res}_{z_1=0} f(z_1) = \frac{1}{2}A_k(0) - \frac{1}{4}A_k(0) + \frac{1}{4}A_k(0)x + \frac{1}{2\log q}A'_k(0). \quad (6.5.11)$$

We find, after some straightforward calculations, that:

$$A_k(0) = P(1) = \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left(1 - \frac{1}{(|P|+1)|P|}\right) \quad (6.5.12)$$

and

$$A'_k(0) = A_k(0)(2\log q) \sum_{\substack{P \text{ monic} \\ \text{irreducible}}} \frac{\deg(P)}{|P|(|P|+1)-1} \quad (6.5.13)$$

and so (6.5.11) becomes

$$\operatorname{Res}_{z_1=0} f(z_1) = \frac{1}{4}P(1) + \frac{1}{4}P(1)x + P(1) \sum_{\substack{P \text{ monic} \\ \text{irreducible}}} \frac{\deg(P)}{|P|(|P|+1)-1}. \quad (6.5.14)$$

Hence we have that,

$$\begin{aligned} & \frac{1}{\pi i} \oint \frac{A_k(\frac{1}{2}; z_1) \zeta_A(1+2z_1) q^{-z_1/2}}{z_1} q^{\frac{x}{2}z_1} dz_1 \\ &= \frac{1}{\pi i} 2\pi i \left( \frac{1}{4}P(1) + \frac{1}{4}P(1)x + P(1) \sum_{\substack{P \text{ monic} \\ \text{irreducible}}} \frac{\deg(P)}{|P|(|P|+1)-1} \right) \\ &= \frac{1}{2}P(1) + \frac{1}{2}P(1)x + 2P(1) \sum_{\substack{P \text{ monic} \\ \text{irreducible}}} \frac{\deg(P)}{|P|(|P|+1)-1}. \end{aligned} \quad (6.5.15)$$

So,

$$Q_1(x) = \frac{1}{2}P(1) \left\{ x + 1 + 4 \sum_{\substack{P \text{ monic} \\ \text{irreducible}}} \frac{\deg(P)}{|P|(|P|+1)-1} \right\}. \quad (6.5.16)$$

We therefore have that

$$\begin{aligned}
\sum_{D \in \mathcal{H}_{2g+1,q}} L\left(\frac{1}{2}, \chi_D\right) &= \sum_{D \in \mathcal{H}_{2g+1,q}} Q_1(\log_q |D|)(1 + O(|D|^{-\frac{1}{2}+\varepsilon})) \\
&= \sum_{D \in \mathcal{H}_{2g+1,q}} \frac{1}{2} P(1) \left\{ \log_q |D| + 1 + 4 \sum_{\substack{P \text{ monic} \\ \text{irreducible}}} \frac{\deg(P)}{|P|(|P|+1)-1} \right\} \\
&\quad \times (1 + O(|D|^{-\frac{1}{2}+\varepsilon})) \\
&= \frac{P(1)}{2} \left\{ \log_q |D| + 1 + 4 \sum_{\substack{P \text{ monic} \\ \text{irreducible}}} \frac{\deg(P)}{|P|(|P|+1)-1} \right\} \sum_{D \in \mathcal{H}_{2g+1,q}} 1 + O(|D|^{\frac{1}{2}+\varepsilon}) \\
&= \frac{P(1)}{2\zeta_A(2)} |D| \left\{ \log_q |D| + 1 + 4 \sum_{\substack{P \text{ monic} \\ \text{irreducible}}} \frac{\deg(P)}{|P|(|P|+1)-1} \right\} + O(|D|^{\frac{1}{2}+\varepsilon}).
\end{aligned}$$

If we compare the main theorem of Chapter 3 with the conjecture we note that the main term and the principal lower order terms are the same. Hence the main theorem of Chapter 3 proves our conjecture with an error  $O\left(|D|^{\frac{3}{4} + \frac{\log_q 2}{2}}\right)$  when  $k = 1$ .

## 6.5.2 Second Moment

For the second moment, Conjecture 6.1.3 asserts that

$$\sum_{D \in \mathcal{H}_{2g+1,q}} L\left(\frac{1}{2}, \chi_D\right)^2 = \sum_{D \in \mathcal{H}_{2g+1,q}} Q_2(\log_q |D|)(1 + O(|D|^{-\frac{1}{2}+\varepsilon})), \quad (6.5.18)$$

where

$$Q_2(x) = \frac{(-1)^2 2^2}{2!} \frac{1}{(2\pi i)^2} \oint \oint \frac{G(z_1, z_2) \Delta(z_1^2, z_2^2)^2}{z_1^3 z_2^3} q^{\frac{x}{2}(z_1+z_2)} dz_1 dz_2. \quad (6.5.19)$$

Denoting by  $A_j(0, \dots, 0)$  the partial derivative, evaluated at zero, of the func-

tion  $A(\frac{1}{2}; z_1, \dots, z_k)$  with respect to  $j$ th variable, we have that,

$$\begin{aligned}
 & \oint \oint \frac{G(z_1, z_2) \Delta(z_1^2, z_2^2)^2}{z_1^3 z_2^3} q^{\frac{x}{2}(z_1+z_2)} dz_1 dz_2 \\
 &= (2\pi i)^2 \left[ \frac{1}{48(\log q)^3} \left( -6A(0, 0)(\log q)^3 - 11xA(0, 0)(\log q)^3 - 6x^2A(0, 0)(\log q)^3 \right. \right. \\
 & - x^3A(0, 0)(\log q)^3 - 11(\log q)^2 A_2(0, 0) - 12x(\log q)^2 A_2(0, 0) \\
 & - 3x^2(\log q)^2 A_2(0, 0) + 2A_{222}(0, 0) - 11(\log q)^2 A_1(0, 0) - 12x(\log q)^2 A_1(0, 0) \\
 & - 3x^2(\log q)^2 A_1(0, 0) - 24(\log q)A_{12}(0, 0) - 12x(\log q)A_{12}(0, 0) - 6A_{122}(0, 0) \\
 & \left. \left. - 6A_{221}(0, 0) + 2A_{111}(0, 0) \right) \right]. \tag{6.5.20}
 \end{aligned}$$

Simplifying we have,

$$\begin{aligned}
 & \oint \oint \frac{G(z_1, z_2) \Delta(z_1^2, z_2^2)^2}{z_1^3 z_2^3} q^{\frac{x}{2}(z_1+z_2)} dz_1 dz_2 \\
 &= (2\pi i)^2 \left[ -\frac{1}{48(\log q)^3} \left( (6 + 11x + 6x^2 + x^3)A(0, 0)(\log q)^3 \right. \right. \\
 & + (11 + 12x + 3x^2)(\log q)^2 (A_2(0, 0) + A_1(0, 0)) + 12(2 + x)(\log q)A_{12}(0, 0) \\
 & \left. \left. - 2(A_{222}(0, 0) - 3A_{122}(0, 0) - 3A_{112}(0, 0) + A_{111}(0, 0)) \right) \right]. \tag{6.5.21}
 \end{aligned}$$

Hence the leading order asymptotic for the second moment for this family of  $L$ -functions can be written, conjecturally, as

$$\sum_{D \in \mathcal{H}_{2g+1, q}} L(\frac{1}{2}, \chi_D)^2 \sim \frac{1}{24\zeta_A(2)} A_k(\frac{1}{2}; 0, 0) |D| (\log_q |D|)^3, \tag{6.5.22}$$

when  $g \rightarrow \infty$ , where

$$A_k(\frac{1}{2}; 0, 0) = \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left( 1 - \frac{4|P|^2 - 3|P| + 1}{|P|^4 + |P|^3} \right). \tag{6.5.23}$$



### 6.5.3 Third Moment

For the third moment, our conjecture states that:

$$\sum_{D \in \mathcal{H}_{2g+1, q}} L(\tfrac{1}{2}, \chi_D)^3 = \sum_{D \in \mathcal{H}_{2g+1, q}} Q_3(\log_q |D|)(1 + O(|D|^{-\frac{1}{2} + \varepsilon})), \quad (6.5.24)$$

where

$$Q_3(x) = \frac{(-1)^{3 \cdot 2^3}}{3!} \frac{1}{(2\pi i)^3} \oint \oint \oint \frac{G(z_1, z_2, z_3) \Delta(z_1^2, z_2^2, z_3^2)^2}{z_1^5 z_2^5 z_3^5} \times q^{\frac{x}{2}(z_1+z_2+z_3)} dz_1 dz_2 dz_3. \quad (6.5.25)$$

Computing the triple contour integral with the help of the symbolic manipulation software MATHEMATICA, we obtain

$$\begin{aligned} & \oint \oint \oint \frac{G(z_1, z_2, z_3) \Delta(z_1^2, z_2^2, z_3^2)^2}{z_1^5 z_2^5 z_3^5} q^{\frac{x}{2}(z_1+z_2+z_3)} dz_1 dz_2 dz_3 \\ &= (2\pi i)^3 \left[ -\frac{1}{11520(\log q)^6} \left( 3(3+x)^2(40+78x+49x^2+12x^3+x^4)A(0, 0, 0)(\log q)^6 \right. \right. \\ &+ 4(471+949x+720x^2+260x^3+45x^4+3x^5)(\log q)^5(A_3(0, 0, 0)+A_2(0, 0, 0) \\ &+ A_1(0, 0, 0))+4(949+1440x+780x^2+180x^3+15x^4)(\log q)^4(A_{23}(0, 0, 0) \\ &+ A_{13}(0, 0, 0)+A_{12}(0, 0, 0))-10(24+26x+9x^2+x^3)(\log q)^3(2A_{333}(0, 0, 0) \\ &- 3A_{233}(0, 0, 0)-3A_{223}(0, 0, 0)+2A_{222}(0, 0, 0)-3A_{133}(0, 0, 0)-36A_{123}(0, 0, 0) \\ &- 3A_{122}(0, 0, 0)-3A_{113}(0, 0, 0)-3A_{112}(0, 0, 0)+2A_{111}(0, 0, 0)) \\ &- 20(26+18x+3x^2)(\log q)^2(A_{2333}(0, 0, 0)+A_{2223}(0, 0, 0) \\ &+ A_{1333}(0, 0, 0)-6A_{1233}(0, 0, 0)-6A_{1223}(0, 0, 0)+A_{1222}(0, 0, 0)-6A_{1123}(0, 0, 0) \\ &+ A_{1113}(0, 0, 0)+A_{1112}(0, 0, 0))+6(3+x)(\log q)(2A_{33333}(0, 0, 0)-5A_{23333}(0, 0, 0) \\ &- 10A_{22333}(0, 0, 0)-10A_{22233}(0, 0, 0)-5A_{22223}(0, 0, 0)+2A_{22222}(0, 0, 0) \\ &- 5A_{13333}(0, 0, 0)+60A_{12233}(0, 0, 0)-5A_{12222}(0, 0, 0)-10A_{11333}(0, 0, 0) \\ &+ 60A_{11233}(0, 0, 0)+60A_{11223}(0, 0, 0)-10A_{11222}(0, 0, 0)-10A_{11133}(0, 0, 0) \\ &- 10A_{11122}(0, 0, 0)-5A_{11113}(0, 0, 0)-5A_{11112}(0, 0, 0)+2A_{11111}(0, 0, 0)) \end{aligned}$$

$$\begin{aligned}
 &+4(3A_{233333}(0,0,0)-20A_{222333}(0,0,0)+3A_{222223}(0,0,0)+3A_{222223}(0,0,0) \\
 &-30A_{123333}(0,0,0)+30A_{122333}(0,0,0)+30A_{122233}(0,0,0)-30A_{122223}(0,0,0) \\
 &+3A_{122222}(0,0,0)+30A_{112333}(0,0,0)+30A_{112223}(0,0,0)-20A_{111333}(0,0,0) \\
 &+30A_{111233}(0,0,0)+30A_{111223}(0,0,0)-20A_{111222}(0,0,0)-30A_{111123} \\
 &+3A_{111113}(0,0,0)+3A_{11112}(0,0,0)) \Big].
 \end{aligned} \tag{6.5.26}$$

And so

$$\sum_{D \in \mathcal{H}_{2g+1,q}} L(\tfrac{1}{2}, \chi_D)^3 \sim \frac{1}{2880\zeta_A(2)} A_k(\tfrac{1}{2}; 0, 0, 0) |D| (\log_q |D|)^6, \tag{6.5.27}$$

as  $g \rightarrow \infty$ , where

$$A_k(\tfrac{1}{2}; 0, 0, 0) = \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left( 1 - \frac{12|P|^5 - 23|P|^4 + 23|P|^3 - 15|P|^2 + 6|P| - 1}{|P|^6(|P| + 1)} \right). \tag{6.5.28}$$

## 6.6 Leading Order Asymptotic for the Moments of $L(s, \chi_D)$

In this section we will show how to obtain an explicit conjecture for the leading order asymptotic of the moments for a general integer  $k$ . The calculations presented here follow closely those presented in [Kea-Odg]. The main result is the following conjecture

**Conjecture 6.6.1.** *Using the same notation as in Conjecture 6.1.3, we have that as  $g \rightarrow \infty$  the following holds*

$$\sum_{D \in \mathcal{H}_{2g+1,q}} L(\tfrac{1}{2}, \chi_D)^k \sim \frac{|D|}{\zeta_A(2)} (\log_q |D|)^{k(k+1)/2} A_k(\tfrac{1}{2}; 0, \dots, 0) \prod_{j=1}^k \frac{j!}{(2j)!}. \tag{6.6.1}$$

To establish the above conjecture we will first prove the following lemma.

**Lemma 6.6.2.** *Suppose  $F$  is a symmetric function of  $k$  variables, regular near  $(0, \dots, 0)$  and  $f(s)$  has a simple pole of residue 1 at  $s = 0$  and is otherwise analytic in a neighborhood of  $s = 0$ . Let*

$$K(|D|; w_1, \dots, w_k) = \sum_{\varepsilon_j = \pm 1} e^{\frac{1}{2} \log |D| \sum_{j=1}^k \varepsilon_j w_j} F(\varepsilon_1 w_1, \dots, \varepsilon_j w_j) \times \prod_{1 \leq i < j \leq k} f(\varepsilon_i w_i + \varepsilon_j w_j), \quad (6.6.2)$$

and define  $I(|D|, k; w = 0)$  to be the value of  $K$  when  $w_1, \dots, w_k = 0$ . We have that,

$$I(|D|, k; 0) \sim \left(\frac{1}{2} \log |D|\right)^{k(k+1)/2} F(0, \dots, 0) 2^{k(k+1)/2} \left( \prod_{j=1}^k \frac{j!}{(2j)!} \right). \quad (6.6.3)$$

*Proof.* We begin by defining the following function

$$G(|D|; w_1, \dots, w_k) = e^{\frac{1}{2} \log |D| \sum_{j=1}^k w_j} F(w_1, \dots, w_k) \prod_{1 \leq i < j \leq k} f(w_i + w_j). \quad (6.6.4)$$

So by Lemma 2.5.2 of [CFKRS05] we have,

$$\begin{aligned} & \sum_{\varepsilon_j = \pm 1} G(|D|; \varepsilon_1 w_1, \dots, \varepsilon_k w_k) \\ &= \frac{(-1)^{k(k-1)/2} 2^k}{(2\pi i)^k k!} \oint \cdots \oint G(|D|; z_1, \dots, z_k) \\ & \quad \times \frac{\Delta(z_1^2, \dots, z_k^2)^2 \prod_{j=1}^k z_j}{\prod_{i=1}^k \prod_{j=1}^k (z_j - w_j)(z_i - w_j)} dz_1 \dots dz_k. \end{aligned} \quad (6.6.5)$$

We will analyze this integral as  $w_j \rightarrow 0$ . It follows from (6.6.5) that

$$\begin{aligned} & I(|D|, k; 0) \\ &= \frac{(-1)^{k(k-1)/2} 2^k}{(2\pi i)^k k!} \oint \cdots \oint G(|D|; z_1, \dots, z_k) \frac{\Delta(z_1^2, \dots, z_k^2)^2 \prod_{j=1}^k z_j}{\prod_{j=1}^k z_j^{2k}} dz_1 \dots dz_k. \end{aligned} \quad (6.6.6)$$

We expand  $G(|D|; z_1, \dots, z_k)$  and make the following variable change  $z_j =$

$\frac{2v_j}{\log|D|}$  which provides us with

$$\begin{aligned}
 & I(|D|, k; 0) \\
 &= \left(\frac{1}{2} \log|D|\right)^{k(k+1)/2} \frac{(-1)^{k(k-1)/2}}{(2\pi i)^k} \frac{1}{k!} \oint \cdots \oint e^{\sum_{j=1}^k v_j} \\
 &\times F\left(\frac{2v_1}{\log|D|}, \dots, \frac{2v_k}{\log|D|}\right) \prod_{1 \leq i < j \leq k} f\left(\frac{2}{\log|D|}(v_i + v_j)\right) \left(\frac{2}{\log|D|}(v_i + v_j)\right) \\
 &\times \prod_{j=1}^k f\left(\frac{2}{\log|D|}(2v_j)\right) \left(\frac{2}{\log|D|}(2v_j)\right) \\
 &\times \prod_{1 \leq i < j \leq k} \frac{1}{v_i + v_j} \frac{\Delta(v_1^2, \dots, v_k^2)^2}{\prod_{j=1}^k v_j^{2k}} dv_1 \dots dv_k.
 \end{aligned} \tag{6.6.7}$$

Letting  $g \rightarrow \infty$  (i.e.  $|D| \rightarrow \infty$ ) we have,

$$\begin{aligned}
 & I(|D|, k; 0) \sim \left(\frac{1}{2} \log|D|\right)^{k(k+1)/2} F(0, \dots, 0) \\
 &\times \frac{(-1)^{k(k-1)/2}}{(2\pi i)^k} \frac{1}{k!} \oint \cdots \oint e^{\sum_{j=1}^k v_j} \prod_{1 \leq i < j \leq k} \frac{1}{v_i + v_j} \frac{\Delta(v_1^2, \dots, v_k^2)^2}{\prod_{j=1}^k v_j^{2k}} dv_1 \dots dv_k.
 \end{aligned} \tag{6.6.8}$$

Using equation (3.36) from [CFKRS03], Lemma 2.5.2 from [CFKRS05], and the second integral of Theorem 1.5.5 for the moments at the symmetry point of characteristic polynomials in the symplectic ensemble completes the proof of the lemma.  $\square$

Now we are ready to establish Conjecture 6.6.1. Using the equation (6.4.9) with  $\alpha_1, \dots, \alpha_k \rightarrow 0$  and the lemma above we have that,

$$\begin{aligned}
 \sum_{D \in \mathcal{H}_{2g+1, q}} L\left(\frac{1}{2}, \chi_D\right)^k &\sim \sum_{D \in \mathcal{H}_{2g+1, q}} \frac{1}{(\log q)^{k(k+1)/2}} \\
 &\times \left(\frac{1}{2} \log|D|\right)^{k(k+1)/2} A_k(0, \dots, 0) 2^{k(k+1/2)} \prod_{j=1}^k \frac{j!}{(2j)!}.
 \end{aligned} \tag{6.6.9}$$

So as  $g \rightarrow \infty$  we have the formula given in the conjecture.

# Chapter 7

## Autocorrelation of Ratios of $L$ -functions over Rational Function Fields $\mathbb{F}_q(T)$

In this chapter we present a heuristic for all of the main terms in the quotient of products of quadratic Dirichlet  $L$ -functions over the rational function field when the average is taken over a family of hyperelliptic curves given by  $C_D : y^2 = D(T)$ , where  $D \in \mathcal{H}_{2g+1,q}$ . The main conjecture presented in this chapter generalize the new Conjecture 6.1.3 presented in the Chapter 6 and can be compared with the the analogous theorem for the characteristic polynomials of matrices averaged over the compact group  $USp(2g)$  and again we find striking similarities between our conjecture and the theorem as expected.

### 7.1 Conrey, Farmer and Zirnbauer's Recipe for Ratios of $L$ -Functions

In [Conr-Far-Zir] we find the following recipe for conjecturing average of ratios of  $L$ -functions.

### 7.1.1 The Recipe

Suppose  $\mathcal{L}$  is an  $L$ -function and  $\mathcal{F} = \{f\}$  is a family of characters with conductor  $c(f)$ , as described in Section 3 of [CFKRS05]. Thus,  $\mathcal{L}(s, f)$  has an approximate functional equation of the form

$$\mathcal{L}(s, f) = \sum \frac{a_n(f)}{n^s} + \varepsilon_f \mathcal{X}_f(s) \sum \frac{\overline{a_n(f)}}{n^{1-s}} + \text{remainder}. \quad (7.1.1)$$

Also, we can write

$$\frac{1}{\mathcal{L}(s, f)} = \sum_{n=1}^{\infty} \frac{\mu_{\mathcal{L}, f}(n)}{n^s}, \quad (7.1.2)$$

the series converging absolutely for  $\Re(s) > 1$  and conditionally, assuming a suitable Riemann Hypothesis, for  $\Re(s) > \frac{1}{2}$ .

We wish to conjecture a precise asymptotic formula for the average

$$\sum_{f \in \mathcal{F}} \frac{\mathcal{L}(\frac{1}{2} + \alpha_1, f) \dots \mathcal{L}(\frac{1}{2} + \alpha_K, f) \mathcal{L}(\frac{1}{2} + \alpha_{K+1}, \overline{f}) \dots \mathcal{L}(\frac{1}{2} + \alpha_{K+L}, \overline{f})}{\mathcal{L}(\frac{1}{2} + \gamma_1, f) \dots \mathcal{L}(\frac{1}{2} + \gamma_Q, f) \mathcal{L}(\frac{1}{2} + \delta_1, \overline{f}) \dots \mathcal{L}(\frac{1}{2} + \delta_R, \overline{f})} g(c(f)) \quad (7.1.3)$$

where  $g$  is a suitable test function. Note that the sum is an integral in the case of moments in  $t$ -aspect.

The recipe is:

1. Start with

$$\begin{aligned} & \mathcal{L}_f(s; \alpha_K; \alpha_L; \gamma_Q; \delta_R) \\ &= \frac{\mathcal{L}(s + \alpha_1, f) \dots \mathcal{L}(s + \alpha_K, f) \mathcal{L}(s + \alpha_{K+1}, \overline{f}) \dots \mathcal{L}(s + \alpha_{K+L}, \overline{f})}{\mathcal{L}(s + \gamma_1, f) \dots \mathcal{L}(s + \gamma_Q, f) \mathcal{L}(s + \delta_1, \overline{f}) \dots \mathcal{L}(s + \delta_R, \overline{f})} \end{aligned} \quad (7.1.4)$$

2. Replace each  $L$ -function in the numerator with the two terms from its approximate functional equation (7.1.1), ignoring the remainder term. Replace each  $L$ -function in the denominator by its series (7.1.2). Multiply out the resulting expression to obtain  $2^{K+L}$  terms. Write those terms as

$$(\text{product of } \varepsilon_f \text{ factors})(\text{product of } \mathcal{X}_f \text{ factors}) \sum_{n_1, \dots, n_{K+L+Q+R}} (\text{summand}). \quad (7.1.5)$$

3. Replace each product of  $\varepsilon_f$ -factors by its expected value when averaged over the family.
4. Replace each summand by its expected value when averaged over the family.
5. Complete the resulting sums (i.e., extend the ranges of the summation indices out to infinity), and call the total  $M_f(s, \alpha_K; \alpha_L; \gamma_Q; \delta_R)$ .
6. The conjecture is

$$\begin{aligned} & \sum_{f \in \mathcal{F}} \mathcal{L}_f\left(\frac{1}{2}, \alpha_K; \alpha_L; \gamma_Q; \delta_R\right) g(c(f)) \\ &= \sum_{f \in \mathcal{F}} M_f\left(\frac{1}{2}, \alpha_K; \alpha_L; \gamma_Q; \delta_R\right) (1 + O(e^{(-\frac{1}{2} + \varepsilon)c(f)})) g(c(f)), \end{aligned} \quad (7.1.6)$$

for all  $\varepsilon > 0$ , where  $g$  is a suitable weight function.

In the rest of this chapter, we will adapt the recipe above for the case of a family of  $L$ -functions associated with hyperelliptic curves over a finite field and in the end we will writing down the function field analogue of the following conjecture.

**Conjecture 7.1.1** (Conrey, Farmer and Zirnbauer). *Suppose that the real parts of  $\alpha_k$  and  $\gamma_q$  are positive. Then*

$$\begin{aligned} & \sum_{0 < d \leq X} \frac{\prod_{k=1}^K L(1/2 + \alpha_k, \chi_d)}{\prod_{q=1}^Q L(1/2 + \gamma_q, \chi_d)} \\ &= \sum_{0 < d \leq X} \sum_{\epsilon \in \{-1, 1\}^K} \left(\frac{|d|}{\pi}\right)^{\frac{1}{2} \sum_{k=1}^K (\epsilon_k \alpha_k - \alpha_k)} \prod_{k=1}^K g_+\left(\frac{1}{2} + \frac{\alpha_k - \epsilon_k \alpha_k}{2}\right) \\ & \quad \times Y_S A_D(\epsilon_1 \alpha_1, \dots, \epsilon_K \alpha_K; \gamma) + O(X^{1/2 + \epsilon}), \end{aligned} \quad (7.1.7)$$

where  $Y_S$  is a certain product of Riemann zeta-functions,  $A_D$  is an Euler product which is absolutely convergent for all of the variables in small disks around 0 and

$$g_+(s) = \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}. \quad (7.1.8)$$

If we let

$$H_{\mathcal{D}^+, d, \alpha, \gamma}(w) = \left(\frac{|d|}{\pi}\right)^{\frac{1}{2} \sum_{k=1}^K w_k} \prod_{k=1}^K g_+ \left(\frac{1}{2} + \frac{\alpha_k - w_k}{2}\right) Y_{SA_{\mathcal{D}}}(w_1, \dots, w_k; \gamma) \quad (7.1.9)$$

then the conjecture may be formulated as

$$\begin{aligned} & \sum_{0 < d \leq X} \frac{\prod_{k=1}^K L(1/2 + \alpha_k, \chi_d)}{\prod_{q=1}^Q L(1/2 + \gamma_q, \chi_d)} \\ &= \sum_{0 < d \leq X} \left(\frac{|d|}{\pi}\right)^{-\frac{1}{2} \sum_{k=1}^K \alpha_k} \sum_{\epsilon \in \{-1, 1\}^K} H_{\mathcal{D}^+, d, \alpha, \gamma}(\epsilon_1 \alpha_1, \dots, \epsilon_K \alpha_K) \\ & \quad + O(X^{1/2+\epsilon}). \end{aligned} \quad (7.1.10)$$

## 7.2 Autocorrelation of Ratios of Characteristic Polynomials

Let  $A \in USp(2N)$ . Thus, the eigenvalues of  $A$  occur in complex conjugate pairs and we can write them as

$$e^{\pm i\theta_1}, \dots, e^{\pm i\theta_N} \quad (7.2.1)$$

with

$$0 \leq \theta_1, \theta_2, \dots, \theta_N \leq \pi. \quad (7.2.2)$$

Now we quote formulas that appear in [Conr-Far-Zir], [Huc-Put-Zir] and [Conr-For-Sna] for the ratios of characteristic polynomials averaged over the symplectic group  $USp(2N)$ . Let

$$z(x) = \frac{1}{1 - e^{-x}} = \frac{1}{x} + O(1), \quad (7.2.3)$$

and note that the function  $z(x)$  appears in random matrix theory where  $\zeta(1+x)$  appears in the study of mean values of  $L$ -functions. Call  $dA$  the Haar measure on the group  $USp(2N)$  and

$$\Lambda_A(s) = \det(I - sA^\dagger), \quad (7.2.4)$$



the characteristic polynomial of  $A$ .

Thus, we have

**Theorem 7.2.1.** *If  $2N \geq Q - K - 1$  and  $\Re(\gamma_q) > 0$  then*

$$\begin{aligned} & \int_{USp(2N)} \frac{\prod_{k=1}^K \Lambda_M(e^{-\alpha_k})}{\prod_{q=1}^Q \Lambda_M(e^{-\gamma_q})} dA \\ &= \sum_{\epsilon \in \{-1,1\}^K} e^{N \sum_{k=1}^K (\epsilon_k \alpha_k - \alpha_k)} \frac{\prod_{j \leq k \leq K} z(\epsilon_j \alpha_j + \epsilon_k \alpha_k) \prod_{q < r \leq Q} z(\gamma_q + \gamma_r)}{\prod_{k=1}^K \prod_{q=1}^Q z(\epsilon_k \alpha_k + \gamma_q)}, \end{aligned} \quad (7.2.5)$$

If we let

$$y_S(\alpha; \gamma) := \frac{\prod_{j \leq k \leq K} z(\alpha_j + \alpha_k) \prod_{q < r \leq Q} z(\gamma_q + \gamma_r)}{\prod_{k=1}^K \prod_{q=1}^Q z(\alpha_k + \gamma_q)} \quad (7.2.6)$$

and

$$h_S(\alpha; \gamma) = e^{N \sum_{k=1}^K \epsilon_k \alpha_k} y_S(\alpha; \gamma), \quad (7.2.7)$$

then the above can be expressed as

$$\int_{USp(2N)} \frac{\prod_{k=1}^K \Lambda_A(e^{-\alpha_k})}{\prod_{q=1}^Q \Lambda_A(e^{-\gamma_q})} dA = e^{-N \sum_{k=1}^K \alpha_k} \sum_{\epsilon \in \{-1,1\}^K} h_S(\epsilon_1 \alpha_1, \dots, \epsilon_K \alpha_K; \gamma). \quad (7.2.8)$$

Note the similarities between Theorem 7.2.1 and the Conjecture 7.1.1.

## 7.3 Applying the Recipe for $L$ -functions over Function Fields.

We will now adapt the recipe presented in Section 7.1.1 for the case of  $L$ -functions over function fields.

Recall that in Chapter 2 we introduced our family of  $L$ -functions. In particular if

$$\mathcal{H}_{2g+1,q} = \{D \text{ monic, } D \text{ square-free, } \deg(D) = 2g + 1, D \in \mathbb{F}_q[x]\}, \quad (7.3.1)$$

the family  $\mathcal{D}^+ = \{L(s, \chi_D) : D \in \mathcal{H}_{2g+1, q}\}$  is a symplectic family. We can make a conjecture which is the function field analogue of the Conjecture 7.1.1 and can be compared with the Theorem 7.2.1 for

$$\sum_{D \in \mathcal{H}_{2g+1, q}} \frac{\prod_{k=1}^K L(\frac{1}{2} + \alpha_k, \chi_D)}{\prod_{q=1}^Q L(\frac{1}{2} + \gamma_q, \chi_D)}. \quad (7.3.2)$$

The main difficulty, like in the number field case, is to identify and factor out the appropriate zeta factors (arithmetic factors). We now follow the recipe given in the section 7.1.1 and we will adapt the recipe for function fields when necessary.

The  $L$ -functions in the numerator are replaced by their “approximate” functional equations

$$L(s, \chi_D) = \sum_{\substack{n \text{ monic} \\ \deg(n) \leq g}} \frac{\chi_D(n)}{|n|^s} + \mathcal{X}_D(s) \sum_{\substack{n \text{ monic} \\ \deg(n) \leq g-1}} \frac{\chi_D(n)}{|n|^{1-s}}, \quad (7.3.3)$$

and those in the denominator are expanded into series

$$\frac{1}{L(s, \chi_D)} = \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left(1 - \frac{\chi_D(P)}{|P|^s}\right) = \sum_{n \text{ monic}} \frac{\mu(n) \chi_D(n)}{|n|^s}, \quad (7.3.4)$$

with  $\mu(n)$  and  $\chi_D(n)$  defined in Chapter 2.

In the numerator we will again replace  $L(s, \chi_D)$  with  $Z_{\mathcal{L}}(s, \chi_D)$  and in the end we will recover the  $L$ -function in the numerator by using that

$$Z_{\mathcal{L}}(s, \chi_D) = \mathcal{X}_D(s)^{-1/2} L(s, \chi_D). \quad (7.3.5)$$

The quantity that we will apply the recipe to is

$$\begin{aligned} & \sum_{D \in \mathcal{H}_{2g+1, q}} \frac{\prod_{k=1}^K Z_{\mathcal{L}}(\frac{1}{2} + \alpha_k, \chi_D)}{\prod_{q=1}^Q L(\frac{1}{2} + \gamma_q, \chi_D)} \\ &= \sum_{D \in \mathcal{H}_{2g+1, q}} Z_{\mathcal{L}}(\frac{1}{2} + \alpha_1, \chi_D) \cdots Z_{\mathcal{L}}(\frac{1}{2} + \alpha_K, \chi_D) \\ & \quad \times \sum_{\substack{h_1, \dots, h_Q \\ h_i \text{ monic}}} \frac{\mu(h_1) \cdots \mu(h_Q) \chi_D(h_1 \cdots h_Q)}{\prod_{q=1}^Q |h_q|^{\frac{1}{2} + \gamma_q}}. \end{aligned} \quad (7.3.6)$$

We have that,

$$\begin{aligned} & Z_{\mathcal{L}}\left(\frac{1}{2} + \alpha_1, \chi_D\right) \dots Z_{\mathcal{L}}\left(\frac{1}{2} + \alpha_K, \chi_D\right) \\ &= \sum_{\epsilon_k \in \{-1, 1\}^K} \prod_{k=1}^K \mathcal{X}_D\left(\frac{1}{2} + \epsilon_k \alpha_k\right)^{-1/2} \sum_{\substack{m_1, \dots, m_K \\ m_j \text{ monic}}} \frac{\chi_D(m_1 \dots m_K)}{\prod_{k=1}^K |m_k|^{\frac{1}{2} + \epsilon_k \alpha_k}}, \end{aligned} \quad (7.3.7)$$

and so, (7.3.6) becomes

$$\begin{aligned} & \sum_{D \in \mathcal{H}_{2g+1, q}} \sum_{\epsilon_k \in \{-1, 1\}^K} \prod_{k=1}^K \mathcal{X}_D\left(\frac{1}{2} + \epsilon_k \alpha_k\right)^{-1/2} \\ & \quad \times \sum_{\substack{m_1, \dots, m_K \\ h_1, \dots, h_Q \\ m_j, h_i \text{ monic}}} \frac{\prod_{q=1}^Q \mu(h_q) \chi_D(m_1 \dots m_K) \chi_D(h_1 \dots h_Q)}{\prod_{k=1}^K |m_k|^{\frac{1}{2} + \epsilon_k \alpha_k} \prod_{q=1}^Q |h_q|^{\frac{1}{2} + \gamma_q}}. \end{aligned} \quad (7.3.8)$$

Now, following the recipe we average the summand over fundamental discriminants  $D \in \mathcal{H}_{2g+1, q}$ ,

$$\begin{aligned} & \lim_{\deg(D) \rightarrow \infty} \left( \sum_{\epsilon_k \in \{-1, 1\}^K} \prod_{k=1}^K \mathcal{X}_D\left(\frac{1}{2} + \epsilon_k \alpha_k\right)^{-1/2} \right. \\ & \quad \times \sum_{\substack{m_1, \dots, m_K \\ h_1, \dots, h_Q \\ m_j, h_i \text{ monic}}} \frac{\prod_{q=1}^Q \mu(h_q) \langle \chi_D(\prod_{k=1}^K m_k \prod_{q=1}^Q h_q) \rangle}{\prod_{k=1}^K |m_k|^{\frac{1}{2} + \epsilon_k \alpha_k} \prod_{q=1}^Q |h_q|^{\frac{1}{2} + \gamma_q}} \left. \right) \\ &= \sum_{\epsilon_k \in \{-1, 1\}^K} \prod_{k=1}^K \mathcal{X}_D\left(\frac{1}{2} + \epsilon_k \alpha_k\right)^{-1/2} \\ & \quad \times \sum_{\substack{m_1, \dots, m_K \\ h_1, \dots, h_Q \\ m_j, h_i \text{ monic}}} \frac{\prod_{q=1}^Q \mu(h_q) \delta\left(\prod_{k=1}^K m_k \prod_{q=1}^Q h_q\right)}{\prod_{k=1}^K |m_k|^{\frac{1}{2} + \epsilon_k \alpha_k} \prod_{q=1}^Q |h_q|^{\frac{1}{2} + \gamma_q}}, \end{aligned} \quad (7.3.9)$$

where  $\delta(n) = \prod_{\substack{P \text{ monic} \\ P \text{ irreducible} \\ P|n}} \left(1 + \frac{1}{|P|}\right)^{-1}$  if  $n$  is a square and is 0 otherwise.

Thus, using the same notation as in [Conr-Far-Zir]

$$G_{\mathcal{D}}(\alpha; \gamma) = \sum_{\substack{m_1, \dots, m_K \\ h_1, \dots, h_Q \\ m_j, h_i \text{ monic}}} \frac{\prod_{q=1}^Q \mu(h_q) \delta \left( \prod_{k=1}^K m_k \prod_{q=1}^Q h_q \right)}{\prod_{k=1}^K |m_k|^{\frac{1}{2} + \alpha_k} \prod_{q=1}^Q |h_q|^{\frac{1}{2} + \gamma_q}}. \quad (7.3.10)$$

We can express  $G_{\mathcal{D}}(\alpha; \gamma)$  as a convergent Euler product provided that  $\Re(\alpha_k) > 0$  and  $\Re(\gamma_q) > 0$ . Thus,

$$G_{\mathcal{D}}(\alpha; \gamma) = \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left( 1 + \left( 1 + \frac{1}{|P|} \right)^{-1} \times \sum_{\substack{0 < \sum_k a_k + \sum_q c_q \\ \text{is even}}} \frac{\prod_{q=1}^Q \mu(P^{c_q})}{|P|^{\sum_k a_k (\frac{1}{2} + \alpha_k) + \sum_q c_q (\frac{1}{2} + \gamma_q)}} \right). \quad (7.3.11)$$

The above expression will enable us to locate the poles and zeros and express  $G_{\mathcal{D}}$  in terms of the zeta-function associated with  $A = \mathbb{F}_q[x]$ . Following Conrey *et.al.* [Conr-Far-Zir] we want to express the contribution of all zeros and poles of the above Euler product in terms of  $\zeta_A(s)$ , doing this we obtain

$$G_{\mathcal{D}}(\alpha; \gamma) = \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left( 1 + \left( 1 + \frac{1}{|P|} \right)^{-1} \left[ \sum_{\substack{j,k \\ j < k}} \frac{1}{|P|^{\left(\frac{1}{2} + \alpha_j\right) + \left(\frac{1}{2} + \alpha_k\right)}} + \sum_k \frac{1}{|P|^{1+2\alpha_k}} + \sum_{\substack{q < r \\ q,r}} \frac{\mu(P)^2}{|P|^{\left(\frac{1}{2} + \gamma_q\right) + \left(\frac{1}{2} + \gamma_r\right)}} + \sum_k \sum_q \frac{\mu(P)}{|P|^{\left(\frac{1}{2} + \alpha_k\right) + \left(\frac{1}{2} + \gamma_q\right)}} + \dots \right] \right), \quad (7.3.12)$$

where  $\dots$  indicates terms that converge. Remembering that,

$$\zeta_A(s) = \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left( 1 - \frac{1}{|P|^s} \right)^{-1} \quad (7.3.13)$$

and using that

$$\left( 1 - \frac{1}{|P|^s} \right)^{-1} = \sum_{j=0}^{\infty} \left( \frac{1}{|P|^s} \right)^j, \quad (7.3.14)$$

we have that the terms in (7.3.12) with  $\sum_{k=1}^K a_k + \sum_{q=1}^Q c_q = 2$  contribute to the zeros and poles. The poles come from terms with  $a_j = a_k = 1$ ,  $1 \leq j < k \leq K$ ,

and from terms  $a_k = 2$ ,  $1 \leq k \leq K$ . In addition, there are poles coming from terms with  $c_q = c_r = 1$ ,  $1 \leq q < r \leq Q$ .

We also note that poles do not arise from terms with  $c_q = 2$  since  $\mu(P^2) = 0$ . The contribution of zeros arises from terms with  $a_k = 1 = c_q$  with  $1 \leq k \leq K$  and  $1 \leq q \leq Q$ . After all this analysis, the contribution, expressed in terms of  $\zeta_A(s)$ , of all these zeros and poles is

$$Y_S(\alpha; \gamma) := \frac{\prod_{j \leq k \leq K} \zeta_A(1 + \alpha_j + \alpha_k) \prod_{q < r \leq Q} \zeta_A(1 + \gamma_q + \gamma_r)}{\prod_{k=1}^K \prod_{q=1}^Q \zeta_A(1 + \alpha_k + \gamma_q)}. \quad (7.3.15)$$

So, when we factor  $Y_S$  out from  $G_{\mathcal{D}}$  we are left with the Euler product  $A_{\mathcal{D}}$  which is absolutely convergent for all of the variables in small disks around 0:

$$\begin{aligned} A_{\mathcal{D}}(\alpha; \gamma) &= \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \frac{\prod_{j \leq k \leq K} \left(1 - \frac{1}{|P|^{1+\alpha_j+\alpha_k}}\right) \prod_{q < r \leq Q} \left(1 - \frac{1}{|P|^{1+\gamma_q+\gamma_r}}\right)}{\prod_{k=1}^K \prod_{q=1}^Q \left(1 - \frac{1}{|P|^{1+\alpha_k+\gamma_q}}\right)} \\ &\quad \times \left(1 + \left(1 + \frac{1}{|P|}\right)^{-1} \sum_{0 < \sum_k a_k + \sum_q c_q \text{ is even}} \frac{\prod_{q=1}^Q \mu(P^{c_q})}{|P|^{\sum_k a_k \left(\frac{1}{2} + \alpha_k\right) + \sum_q c_q \left(\frac{1}{2} + \gamma_q\right)}}\right). \end{aligned} \quad (7.3.16)$$

So we can conclude that,

$$\begin{aligned} \sum_{D \in \mathcal{H}_{2g+1,q}} \frac{\prod_{k=1}^K Z_{\mathcal{L}}\left(\frac{1}{2} + \alpha_k, \chi_D\right)}{\prod_{q=1}^Q L\left(\frac{1}{2} + \gamma_q, \chi_D\right)} &= \sum_{D \in \mathcal{H}_{2g+1,q}} \sum_{\epsilon \in \{-1,1\}^K} \prod_{k=1}^K \mathcal{X}_D\left(\frac{1}{2} + \epsilon_k \alpha_k\right)^{-1/2} \\ &\quad \times Y_S(\epsilon_1 \alpha_1, \dots, \epsilon_K \alpha_K; \gamma) A_{\mathcal{D}}(\epsilon_1 \alpha_1, \dots, \epsilon_K \alpha_K; \gamma) + O(|D|^{1/2+\epsilon}), \end{aligned} \quad (7.3.17)$$

using (7.3.5) we have that,

$$\begin{aligned} \sum_{D \in \mathcal{H}_{2g+1,q}} \frac{\prod_{k=1}^K L\left(\frac{1}{2} + \alpha_k, \chi_D\right)}{\prod_{q=1}^Q L\left(\frac{1}{2} + \gamma_q, \chi_D\right)} &= \sum_{D \in \mathcal{H}_{2g+1,q}} \sum_{\epsilon \in \{-1,1\}^K} \prod_{k=1}^K \mathcal{X}_D\left(\frac{1}{2} + \epsilon_k \alpha_k\right)^{-1/2} \prod_{k=1}^K \mathcal{X}_D\left(\frac{1}{2} + \alpha_k\right)^{1/2} \\ &\quad \times Y_S(\epsilon_1 \alpha_1, \dots, \epsilon_K \alpha_K; \gamma) A_{\mathcal{D}}(\epsilon_1 \alpha_1, \dots, \epsilon_K \alpha_K; \gamma) + O(|D|^{1/2+\epsilon}), \end{aligned} \quad (7.3.18)$$

moreover,

$$\mathcal{X}_D\left(\frac{1}{2} + \epsilon_k \alpha_k\right)^{-1/2} = |D|^{\frac{1}{2} \epsilon_k \alpha_k} \mathcal{X}\left(\frac{1}{2} + \epsilon_k \alpha_k\right)^{-1/2} \quad (7.3.19)$$

and

$$\mathcal{X}_D(\tfrac{1}{2} + \alpha_k)^{1/2} = |D|^{-\frac{1}{2}\alpha_k} \mathcal{X}(\tfrac{1}{2} + \alpha_k)^{1/2}, \quad (7.3.20)$$

and so

$$\begin{aligned} & \prod_{k=1}^K \mathcal{X}_D(\tfrac{1}{2} + \epsilon_k \alpha_k)^{-1/2} \mathcal{X}_D(\tfrac{1}{2} + \alpha_k)^{1/2} \\ &= \prod_{k=1}^K |D|^{\frac{1}{2}(\epsilon_k \alpha_k - \alpha_k)} \prod_{k=1}^K \mathcal{X}(\tfrac{1}{2} + \epsilon_k \alpha_k)^{-1/2} \mathcal{X}(\tfrac{1}{2} + \alpha_k)^{1/2} \\ &= |D|^{\frac{1}{2} \sum_{k=1}^K (\epsilon_k \alpha_k - \alpha_k)} \prod_{k=1}^K \mathcal{X}(\tfrac{1}{2} + \epsilon_k \alpha_k)^{-1/2} \mathcal{X}(\tfrac{1}{2} + \alpha_k)^{1/2}. \end{aligned} \quad (7.3.21)$$

To put our conjecture in the same form as conjecture 5.2 in [Conr-Far-Zir] and see clearly the analogies between the conjectures for the classical quadratic  $L$ -functions and the  $L$ -functions over function fields, we need first to establish the following simple lemma

**Lemma 7.3.1.** *We have that,*

$$\mathcal{X}\left(\tfrac{1}{2} + \epsilon_k \alpha_k\right)^{-1/2} \mathcal{X}\left(\tfrac{1}{2} + \alpha_k\right)^{1/2} = \mathcal{X}\left(\tfrac{1}{2} + \frac{\alpha_k - \epsilon_k \alpha_k}{2}\right). \quad (7.3.22)$$

*Proof.* Follows directly from the  $\mathcal{X}(s) = q^{-1/2+s}$ .  $\square$

We are now in a position to formulate the desired conjecture for function fields.

**Conjecture 7.3.2.** *Suppose that the real parts of  $\alpha_k$  and  $\gamma_q$  are positive and that  $q \equiv 1 \pmod{4}$  is the fixed cardinality of the finite field  $\mathbb{F}_q$ . Then we have,*

$$\begin{aligned} & \sum_{D \in \mathcal{H}_{2g+1,q}} \frac{\prod_{k=1}^K L(\tfrac{1}{2} + \alpha_k, \chi_D)}{\prod_{q=1}^Q L(\tfrac{1}{2} + \gamma_q, \chi_D)} \\ &= \sum_{D \in \mathcal{H}_{2g+1,q}} \sum_{\epsilon \in \{-1,1\}^K} |D|^{\frac{1}{2} \sum_{k=1}^K (\epsilon_k \alpha_k - \alpha_k)} \prod_{k=1}^K \mathcal{X}\left(\tfrac{1}{2} + \frac{\alpha_k - \epsilon_k \alpha_k}{2}\right) \\ & \quad \times Y_S(\epsilon_1 \alpha_1, \dots, \epsilon_K \alpha_K; \gamma) A_{\mathcal{D}}(\epsilon_1 \alpha_1, \dots, \epsilon_K \alpha_K; \gamma) + O(|D|^{1/2+\epsilon}). \end{aligned} \quad (7.3.23)$$

If we let,

$$H_{\mathcal{D},|D|,\alpha,\gamma}(w) = |D|^{\frac{1}{2}\sum_{k=1}^K w_k} \prod_{k=1}^K \mathcal{X}\left(\frac{1}{2} + \frac{\alpha_k - w_k}{2}\right) \times Y_S(w_1, \dots, w_K; \gamma) A_{\mathcal{D}}(w_1, \dots, w_K; \gamma) \quad (7.3.24)$$

then the conjecture may be formulated as

$$\begin{aligned} & \sum_{D \in \mathcal{H}_{2g+1,q}} \frac{\prod_{k=1}^K L(\frac{1}{2} + \alpha_k, \chi_D)}{\prod_{q=1}^Q L(\frac{1}{2} + \gamma_q, \chi_D)} \\ &= \sum_{D \in \mathcal{H}_{2g+1,q}} |D|^{-\frac{1}{2}\sum_{k=1}^K \alpha_k} \sum_{\epsilon \in \{-1,1\}^K} H_{\mathcal{D},|D|,\alpha,\gamma}(\epsilon_1 \alpha_1, \dots, \epsilon_K \alpha_K; \gamma) + O(|D|^{1/2+\epsilon}). \end{aligned} \quad (7.3.25)$$

**Remark 7.3.3.** Note that the formulas (7.3.23) and (7.3.25) can be seen as the function field analogues of the formulae (5.27) and (5.29) in [Conr-Far-Zir].

## 7.4 Refinements of the Conjecture

In this section we refine the ratios conjecture first by deriving a closed form expression for the Euler product  $A_{\mathcal{D}}(\alpha; \gamma)$ , and second by expressing the combinatorial sum as a multiple integral. This is similar to the treatment given in the previous chapter.

### 7.4.1 Closed form expression for $A_{\mathcal{D}}$

Suppose that  $f(x) = 1 + \sum_{n=1}^{\infty} u_n x^n$ . Then

$$\sum_{0 < n \text{ is even}} u_n x^n = \frac{1}{2}(f(x) + f(-x) - 2) \quad (7.4.1)$$

and so,

$$\begin{aligned}
 1 + \left(1 + \frac{1}{|P|}\right)^{-1} \sum_{0 < n \text{ is even}} u_n x^n &= 1 + \left(1 + \frac{1}{|P|}\right)^{-1} \left(\frac{1}{2}(f(x) + f(-x) - 2)\right) \\
 &= \frac{1}{1 + \frac{1}{|P|}} \left(\frac{f(x) + f(-x)}{2} + \frac{1}{|P|}\right). \tag{7.4.2}
 \end{aligned}$$

Now, let

$$\begin{aligned}
 f\left(\frac{1}{|P|}\right) &= \sum_{a_k, c_q} \frac{\prod_{q=1}^Q \mu(P^{c_q})}{|P|^{\sum_k a_k (\frac{1}{2} + \alpha_k) + \sum_q c_q (\frac{1}{2} + \gamma_q)}} \\
 &= \sum_{a_k} \frac{1}{|P|^{\sum_k a_k (\frac{1}{2} + \alpha_k)}} \sum_{c_q} \frac{\prod_{q=1}^Q \mu(P^{c_q})}{|P|^{\sum_q c_q (\frac{1}{2} + \gamma_q)}} \\
 &= \sum_{a_k} \prod_{k=1}^K \frac{1}{|P|^{a_k (\frac{1}{2} + \alpha_k)}} \sum_{c_q} \prod_{q=1}^Q \frac{\mu(P^{c_q})}{|P|^{c_q (\frac{1}{2} + \gamma_q)}} \\
 &= \frac{\prod_{q=1}^Q \left(1 - \frac{1}{|P|^{1/2 + \gamma_q}}\right)}{\prod_{k=1}^K \left(1 - \frac{1}{|P|^{1/2 + \alpha_k}}\right)}. \tag{7.4.3}
 \end{aligned}$$

We are ready to prove the following lemma

**Lemma 7.4.1.** *We have that,*

$$\begin{aligned}
 1 + \left(1 + \frac{1}{|P|}\right)^{-1} \sum_{0 < \sum_k a_k + \sum_q c_q \text{ is even}} \frac{\prod_{q=1}^Q \mu(P^{c_q})}{|P|^{\sum_k a_k (\frac{1}{2} + \alpha_k) + \sum_q c_q (\frac{1}{2} + \gamma_q)}} \\
 = \frac{1}{1 + \frac{1}{|P|}} \left( \frac{1}{2} \frac{\prod_{q=1}^Q \left(1 - \frac{1}{|P|^{1/2 + \gamma_q}}\right)}{\prod_{k=1}^K \left(1 - \frac{1}{|P|^{1/2 + \alpha_k}}\right)} + \frac{1}{2} \frac{\prod_{q=1}^Q \left(1 + \frac{1}{|P|^{1/2 + \gamma_q}}\right)}{\prod_{k=1}^K \left(1 + \frac{1}{|P|^{1/2 + \alpha_k}}\right)} + \frac{1}{|P|} \right). \tag{7.4.4}
 \end{aligned}$$

*Proof.* The proof follows directly using (7.4.2) and (7.4.3).  $\square$

We have the following corollary from this lemma



**Corollary 7.4.2.**

$$\begin{aligned}
 A_{\mathcal{D}}(\alpha; \gamma) &= \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \frac{\prod_{j \leq k \leq K} \left(1 - \frac{1}{|P|^{1+\alpha_j+\alpha_k}}\right) \prod_{q < r \leq Q} \left(1 - \frac{1}{|P|^{1+\gamma_q+\gamma_r}}\right)}{\prod_{k=1}^K \prod_{q=1}^Q \left(1 - \frac{1}{|P|^{1+\alpha_k+\gamma_q}}\right)} \\
 &\times \frac{1}{1 + \frac{1}{|P|}} \left( \frac{1}{2} \frac{\prod_{q=1}^Q \left(1 - \frac{1}{|P|^{1/2+\gamma_q}}\right)}{\prod_{k=1}^K \left(1 - \frac{1}{|P|^{1/2+\alpha_k}}\right)} + \frac{1}{2} \frac{\prod_{q=1}^Q \left(1 + \frac{1}{|P|^{1/2+\gamma_q}}\right)}{\prod_{k=1}^K \left(1 + \frac{1}{|P|^{1/2+\alpha_k}}\right)} + \frac{1}{|P|} \right).
 \end{aligned} \tag{7.4.5}$$

## 7.4.2 Combinatorial Sum as Multiple Integrals

We begin this subsection by quoting the following lemma from [Conr-Far-Zir].

**Lemma 7.4.3.** *Suppose that  $F(z) = F(z_1, \dots, z_K)$  is a function of  $K$  variables, which is symmetric and regular near  $(0, \dots, 0)$ . Suppose further that  $f(s)$  has a simple pole of residue 1 at  $s = 0$  but is otherwise analytic in  $|s| \leq 1$ . Let either*

$$H(z_1, \dots, z_K) = F(z_1, \dots, z_K) \prod_{1 \leq j \leq k \leq K} f(z_j + z_k) \tag{7.4.6}$$

or

$$H(z_1, \dots, z_K) = F(z_1, \dots, z_K) \prod_{1 \leq j < k \leq K} f(z_j + z_k). \tag{7.4.7}$$

If  $|\alpha_k| < 1$  then

$$\begin{aligned}
 &\sum_{\epsilon \in \{-1, +1\}^K} H(\epsilon_1 \alpha_1, \dots, \epsilon_K \alpha_K) \\
 &= \frac{(-1)^{K(K-1)/2} 2^K}{K! (2\pi i)^K} \int_{|z_i|=1} \frac{H(z_1, \dots, z_K) \Delta(z_1^2, \dots, z_K^2)^2 \prod_{k=1}^K z_k}{\prod_{j=1}^K \prod_{k=1}^K (z_k - \alpha_j)(z_k + \alpha_j)} dz_1 \dots dz_K
 \end{aligned} \tag{7.4.8}$$

and

$$\begin{aligned}
 &\sum_{\epsilon \in \{-1, +1\}^K} \text{sgn}(\epsilon) H(\epsilon_1 \alpha_1, \dots, \epsilon_K \alpha_K) \\
 &= \frac{(-1)^{K(K-1)/2} 2^K}{K! (2\pi i)^K} \int_{|z_i|=1} \frac{H(z_1, \dots, z_K) \Delta(z_1^2, \dots, z_K^2)^2 \prod_{k=1}^K \alpha_k}{\prod_{j=1}^K \prod_{k=1}^K (z_k - \alpha_j)(z_k + \alpha_j)} dz_1 \dots dz_K.
 \end{aligned} \tag{7.4.9}$$

Using this Lemma, we can reformulate Theorem 7.2.1 as

$$\int_{USp(2N)} \frac{\prod_{k=1}^K \Lambda_A(e^{-\alpha_k})}{\prod_{q=1}^Q \Lambda_A(e^{-\gamma_q})} dA = e^{-\frac{N}{2} \sum_{k=1}^K \alpha_k} \frac{(-1)^{K(K-1)/2} 2^K}{K!(2\pi i)^K} \\ \times \int_{|z_i|=1} \frac{h_S(z_1, \dots, z_K; \gamma) \Delta(z_1^2, \dots, z_K^2)^2 \prod_{k=1}^K z_k}{\prod_{j=1}^K \prod_{k=1}^K (z_k - \alpha_j)(z_k + \alpha_j)} dz_1 \dots dz_K. \quad (7.4.10)$$

## 7.5 The Final Form of the Conjecture

Now we are in a position to present the final form of the ratios conjecture for  $L$ -functions over function fields using the contour integrals introduced above. Our main Conjecture 7.3.2 in this section can be written as follows.

**Conjecture 7.5.1.** *Suppose that the real parts of  $\alpha_k$  and  $\gamma_q$  are positive. Then*

$$\sum_{D \in \mathcal{H}_{2g+1,q}} \frac{\prod_{k=1}^K L(\frac{1}{2} + \alpha_k, \chi_D)}{\prod_{q=1}^Q L(\frac{1}{2} + \gamma_q, \chi_D)} = \sum_{D \in \mathcal{H}_{2g+1,q}} |D|^{-\frac{1}{2} \sum_{k=1}^K \alpha_k} \frac{(-1)^{K(K-1)/2} 2^K}{K!(2\pi i)^K} \\ \times \int_{|z_i|=1} \frac{H_{\mathcal{D}, |D|, \alpha, \gamma}(z_1, \dots, z_K; \gamma) \Delta(z_1^2, \dots, z_K^2)^2 \prod_{k=1}^K z_k}{\prod_{j=1}^K \prod_{k=1}^K (z_k - \alpha_j)(z_k + \alpha_j)} dz_1 \dots dz_K \\ + O(|D|^{1/2+\varepsilon}). \quad (7.5.1)$$

**Remark 7.5.2.** *If we compare the formula (7.5.1) with the formula (6.31) presented in [Conr-Far-Zir] we can see clearly the analogy between the classical conjecture and its translation for function fields.*

# Chapter 8

## An Application of the Ratios Conjecture of $L$ -functions over Function Fields: One-Level Density

In this chapter we present an application of the Ratios Conjecture for  $L$ -functions over function fields: we derive a smooth linear statistic, the *one-level density*. The ideas and calculations presented in this chapter can be seen as a translation to the function fields language of the calculations presented in [Conr-Sna] and [Huy-Kea-Sna].

### 8.1 Applying the Ratios Recipe

We present the calculations in full to illustrate the steps outlined in the previous chapter. Our goal is to obtain an asymptotic formula for

$$R_D(\alpha; \gamma) = \sum_{D \in \mathcal{H}_{2g+1, q}} \frac{L(\frac{1}{2} + \alpha, \chi_D)}{L(\frac{1}{2} + \gamma, \chi_D)}. \quad (8.1.1)$$

Following the recipe presented in the Chapter 7, we represent the  $L(s, \chi_D)$

in the numerator by

$$L\left(\frac{1}{2} + \alpha, \chi_D\right) = \sum_{\substack{m \text{ monic} \\ \deg(m) \leq g}} \frac{\chi_D(m)}{|m|^{1/2+\alpha}} + |D|^{-\alpha} \mathcal{X}\left(\frac{1}{2} + \alpha\right) \sum_{\substack{n \text{ monic} \\ \deg(n) \leq g-1}} \frac{\chi_D(n)}{|n|^{1/2-\alpha}}, \quad (8.1.2)$$

and we replace the the  $L(s, \chi_D)$  in the denominator by

$$\frac{1}{L(s, \chi_D)} = \sum_{h \text{ monic}} \frac{\mu(h) \chi_D(h)}{|h|^s}. \quad (8.1.3)$$

When we take the average over the family  $\mathcal{H}_{2g+1,q}$ , we retain only the terms over squares; in other words, we use the first part of the formula

$$\sum_{D \in \mathcal{H}_{2g+1,q}} \chi_D(n) = \begin{cases} a(n) \#\mathcal{H}_{2g+1,q} + \text{small} & \text{if } n \text{ is a square,} \\ \text{small} & \text{if } n \text{ is not a square,} \end{cases} \quad (8.1.4)$$

where

$$a(n) = \prod_{\substack{P \text{ monic} \\ \text{irreducible} \\ P|n}} \frac{|P|}{|P| + 1}, \quad (8.1.5)$$

and

$$\#\mathcal{H}_{2g+1,q} = \sum_{D \in \mathcal{H}_{2g+1,q}} 1 = \frac{|D|}{\zeta_A(2)}. \quad (8.1.6)$$

Now we compute the square terms and complete the sums by extending the range of summation to all monic polynomials. After that, we need to identify the terms which are ratios of products of zeta-functions associated to  $A = \mathbb{F}_q[x]$  (the divergent part), which are multiplied by an absolutely convergent Euler product. We do this for each piece of the ‘‘approximate’’ functional equation to obtain our conjectural result for  $R_D(\alpha; \gamma)$ .

We now present the details involved in the calculation of  $R_D(\alpha; \gamma)$ .

$$\begin{aligned} R_D(\alpha; \gamma) &= \sum_{D \in \mathcal{H}_{2g+1,q}} \sum_{\substack{m \text{ monic} \\ \deg(m) \leq g}} \sum_{h \text{ monic}} \frac{\chi_D(m) \mu(h) \chi_D(h)}{|m|^{1/2+\alpha} |h|^{1/2+\gamma}} \\ &+ \sum_{D \in \mathcal{H}_{2g+1,q}} |D|^{-\alpha} \mathcal{X}\left(\frac{1}{2} + \alpha\right) \sum_{\substack{n \text{ monic} \\ \deg(n) \leq g-1}} \sum_{h \text{ monic}} \frac{\chi_D(n) \mu(h) \chi_D(h)}{|n|^{1/2-\alpha} |h|^{1/2+\gamma}}. \end{aligned} \quad (8.1.7)$$

We focus in the first piece of the “approximate” functional equation. Thus we consider

$$\sum_{D \in \mathcal{H}_{2g+1,q}} \sum_{m,h \text{ monic}} \frac{\mu(h) \chi_D(mh)}{|m|^{\frac{1}{2}+\alpha} |h|^{\frac{1}{2}+\gamma}} = \sum_{m,h \text{ monic}} \frac{\mu(h)}{|m|^{\frac{1}{2}+\alpha} |h|^{\frac{1}{2}+\gamma}} \sum_{D \in \mathcal{H}_{2g+1,q}} \chi_D(mh). \quad (8.1.8)$$

Retaining only the terms for which  $hm = \square$ , lead us to

$$\#\mathcal{H}_{2g+1,q} \sum_{\substack{m,h \text{ monic} \\ hm = \square = j^2}} \frac{\mu(h)}{|m|^{\frac{1}{2}+\alpha} |h|^{\frac{1}{2}+\gamma}} a(hm). \quad (8.1.9)$$

We will express this sum as an Euler product in the following way,

$$\begin{aligned} & \sum_{\substack{m,h \text{ monic} \\ hm = \square = j^2}} \frac{\mu(h)}{|m|^{\frac{1}{2}+\alpha} |h|^{\frac{1}{2}+\gamma}} a(hm) = \sum_j \sum_{\substack{m,h \text{ monic} \\ hm = \square = j^2}} \frac{\mu(h)}{|m|^{\frac{1}{2}+\alpha} |h|^{\frac{1}{2}+\gamma}} a(hm) \\ & = \sum_j a(j^2) \sum_{\substack{m,h \text{ monic} \\ hm = \square = j^2}} \frac{\mu(h)}{|m|^{\frac{1}{2}+\alpha} |h|^{\frac{1}{2}+\gamma}}. \end{aligned} \quad (8.1.10)$$

Let

$$\psi(j^2) = \sum_{\substack{m,h \text{ monic} \\ hm = \square = j^2}} \frac{\mu(h)}{|m|^{\frac{1}{2}+\alpha} |h|^{\frac{1}{2}+\gamma}}, \quad (8.1.11)$$

which is multiplicative, we also have that  $a(j^2)$  is multiplicative. So in the end we obtain that,

$$\sum_j a(j^2) \psi(j^2) = \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left( 1 + \sum_{\nu=1}^{\infty} a(P^{2\nu}) \psi(P^{2\nu}) \right). \quad (8.1.12)$$

And once  $hm = P^{2\nu}$  we have that  $h = P^{e_1}$  and  $m = P^{e_2}$  and  $e_1 + e_2 = 2\nu$ , we can write

$$\psi(P^{2\nu}) = \sum_{\substack{e_1, e_2 \geq 0 \\ e_1 + e_2 = 2\nu}} \frac{\mu(P^{e_1})}{|P|^{e_1(\frac{1}{2}+\gamma)} |P|^{e_2(\frac{1}{2}+\alpha)}}. \quad (8.1.13)$$

And so we can write (8.1.12) as,

$$\begin{aligned} & \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left( 1 + \sum_{\nu=1}^{\infty} a(P^{2\nu}) \sum_{\substack{e_1, e_2 \geq 0 \\ e_1 + e_2 = 2\nu}} \frac{\mu(P^{e_1})}{|P|^{e_1(\frac{1}{2}+\gamma)} |P|^{e_2(\frac{1}{2}+\alpha)}} \right) \\ & = \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left( \sum_{\substack{e_1, e_2 \geq 0 \\ e_1 + e_2 = 2\nu}} \frac{\mu(P^{e_1}) a(P^{e_1+e_2})}{|P|^{e_1(\frac{1}{2}+\gamma)} |P|^{e_2(\frac{1}{2}+\alpha)}} \right). \end{aligned} \quad (8.1.14)$$

The effect of  $\mu(P^{e_1})$  is to limit the choices for  $e_1$  to 0 or 1. When  $e_1 = 0$  we have,

$$\begin{aligned}
 \sum_{e_2 \text{ even}} \frac{a(P^{e_2})}{|P|^{e_2(1/2+\alpha)}} &= \sum_{e_2=0}^{\infty} \frac{a(P^{2e_2})}{|P|^{e_2(1+2\alpha)}} \\
 &= 1 + \sum_{e_2=1}^{\infty} \frac{a(P^{2e_2})}{|P|^{e_2(1+2\alpha)}} \\
 &= 1 + \frac{|P|}{|P|+1} \sum_{e_2=1}^{\infty} \frac{1}{|P|^{e_2(1+2\alpha)}} \\
 &= 1 + \frac{|P|}{|P|+1} \frac{1}{|P|^{1+2\alpha}} \frac{1}{\left(1 - \frac{1}{|P|^{1+2\alpha}}\right)}. \quad (8.1.15)
 \end{aligned}$$

And when  $e_1 = 1$  there is a contribution of

$$\begin{aligned}
 \sum_{e_2 \text{ odd}} \frac{\mu(P)a(P^{2+2e_2})}{|P|^{(1/2+\gamma)+(2e_2+1)(1/2+\alpha)}} &= - \sum_{\substack{e_2=0 \\ e_2 \text{ odd}}}^{\infty} \frac{a(P^{1+e_2})}{|P|^{(1/2+\gamma)+e_2((1/2+\alpha))}} \\
 &= - \sum_{e_2=0}^{\infty} \frac{a(P^{2+2e_2})}{|P|^{1/2+\gamma}|P|^{e_2(1+2\alpha)+(1/2+\alpha)}} \\
 &= - \frac{|P|}{|P|+1} \frac{1}{|P|^{1+\alpha+\gamma}} \sum_{e_2=0}^{\infty} \left(\frac{1}{|P|^{1+2\alpha}}\right)^{e_2} \\
 &= - \frac{|P|}{|P|+1} \frac{1}{|P|^{1+\alpha+\gamma}} \frac{1}{\left(1 - \frac{1}{|P|^{1+2\alpha}}\right)}. \quad (8.1.16)
 \end{aligned}$$

Hence, the Euler product (8.1.14) simplifies to

$$\prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left( 1 + \frac{|P|}{|P|+1} \frac{1}{|P|^{1+2\alpha}} \frac{1}{\left(1 - \frac{1}{|P|^{1+2\alpha}}\right)} - \frac{|P|}{|P|+1} \frac{1}{|P|^{1+\alpha+\gamma}} \frac{1}{\left(1 - \frac{1}{|P|^{1+2\alpha}}\right)} \right). \quad (8.1.17)$$

And we can factor out the appropriate  $\zeta_A$ -factors and write as,

$$\begin{aligned}
 & \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left( 1 + \frac{|P|}{|P|+1} \frac{1}{|P|^{1+2\alpha}} \frac{1}{\left(1 - \frac{1}{|P|^{1+2\alpha}}\right)} \right. \\
 & \qquad \qquad \qquad \left. - \frac{|P|}{|P|+1} \frac{1}{|P|^{1+\alpha+\gamma}} \frac{1}{\left(1 - \frac{1}{|P|^{1+2\alpha}}\right)} \right) \\
 &= \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left( 1 - \frac{1}{|P|^{1+2\alpha}} \right)^{-1} \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left( 1 - \frac{1}{|P|^{1+\alpha+\gamma}} \right) \\
 &\times \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left( 1 - \frac{1}{|P|^{1+\alpha+\gamma}} \right)^{-1} \left( 1 - \frac{1}{|P|^{1+2\alpha}} + \frac{|P|}{(|P|+1)} \frac{1}{|P|^{1+2\alpha}} \right. \\
 & \qquad \qquad \qquad \left. - \frac{|P|}{(|P|+1)} \frac{1}{|P|^{1+\alpha+\gamma}} \right) \\
 &= \frac{\zeta_A(1+2\alpha)}{\zeta_A(1+\alpha+\gamma)} \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left( 1 - \frac{1}{|P|^{1+\alpha+\gamma}} \right)^{-1} \\
 & \qquad \qquad \qquad \times \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left( 1 - \frac{1}{(|P|+1)|P|^{1+2\alpha}} - \frac{1}{(|P|+1)|P|^{\alpha+\gamma}} \right).
 \end{aligned} \tag{8.1.18}$$

The product over “prime” polynomials  $P$  is absolutely convergent as long as  $\Re(\alpha), \Re(\gamma) > -1/4$ .

For the second piece of the “approximate” functional equation, we can determine by recalling the functional equation given by

$$L\left(\frac{1}{2} + \alpha, \chi_D\right) = |D|^{-\alpha} \mathcal{X}\left(\frac{1}{2} + \alpha\right) L\left(\frac{1}{2} - \alpha, \chi_D\right). \tag{8.1.19}$$

Thus, in total, we expect that following conjecture is true

**Conjecture 8.1.1.** *With  $-\frac{1}{4} < \Re(\alpha) < \frac{1}{4}$ ,  $\frac{1}{\log|D|} \ll \Re(\gamma) < \frac{1}{4}$  and  $\Im(\alpha), \Im(\gamma) \ll_\varepsilon$*

$|D|^{1-\varepsilon}$  for every  $\varepsilon > 0$ , we have

$$\begin{aligned} R_D(\alpha; \gamma) &= \sum_{D \in \mathcal{H}_{2g+1, q}} \frac{L(\frac{1}{2} + \alpha, \chi_D)}{L(\frac{1}{2} + \gamma, \chi_D)} \\ &= \sum_{D \in \mathcal{H}_{2g+1, q}} \left( \frac{\zeta_A(1 + 2\alpha)}{\zeta_A(1 + \alpha + \gamma)} A_D(\alpha; \gamma) \right. \\ &\quad \left. + |D|^{-\alpha} \mathcal{X}(\frac{1}{2} + \alpha) \frac{\zeta_A(1 - 2\alpha)}{\zeta_A(1 - \alpha + \gamma)} A_D(-\alpha; \gamma) \right) + O(|D|^{1/2+\varepsilon}), \end{aligned} \quad (8.1.20)$$

where

$$\begin{aligned} A_D(\alpha; \gamma) &= \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left( 1 - \frac{1}{|P|^{1+\alpha+\gamma}} \right)^{-1} \\ &\quad \times \left( 1 - \frac{1}{(|P| + 1)|P|^{1+2\alpha}} - \frac{1}{(|P| + 1)|P|^{\alpha+\gamma}} \right). \end{aligned} \quad (8.1.21)$$

## 8.2 Mean Value Theorem for the Logarithmic Derivative of $L(s, \chi_D)$

To obtain the formula for the one-level density from the ratios conjecture, we note that

$$\sum_{D \in \mathcal{H}_{2g+1, q}} \frac{L'(\frac{1}{2} + r, \chi_D)}{L(\frac{1}{2} + r, \chi_D)} = \frac{d}{d\alpha} R_D(\alpha; \gamma) \Big|_{\alpha=\gamma=r}. \quad (8.2.1)$$

Now, a straightforward calculation gives us

$$\frac{d}{d\alpha} \frac{\zeta_A(1 + 2\alpha)}{\zeta_A(1 + \alpha + \gamma)} A_D(\alpha; \gamma) \Big|_{\alpha=\gamma=r} = \frac{\zeta'_A(1 + 2r)}{\zeta_A(1 + 2r)} A_D(r; r) + A'_D(r; r), \quad (8.2.2)$$

and a simple use of the quotient rule gives us the following formula

$$\begin{aligned} \frac{d}{d\alpha} \left( |D|^{-\alpha} \mathcal{X}(\frac{1}{2} + \alpha) \frac{\zeta_A(1 - 2\alpha)}{\zeta_A(1 - \alpha + \gamma)} A_D(-\alpha; \gamma) \right) \Big|_{\alpha=\gamma=r} \\ = -(\log q) |D|^{-r} \mathcal{X}(\frac{1}{2} + r) \zeta_A(1 - 2r) A_D(-r; r). \end{aligned} \quad (8.2.3)$$



Also,

$$A_D(r; r) = 1, \tag{8.2.4}$$

$$A_D(-r; r) = \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left(1 - \frac{1}{|P|}\right)^{-1} \left(1 - \frac{1}{(|P|+1)|P|^{1-2r}} - \frac{1}{(|P|+1)}\right), \tag{8.2.5}$$

and computing the logarithmic–derivative we can easily obtain that

$$A'_D(r; r) = \sum_{\substack{P \text{ monic} \\ \text{irreducible}}} \frac{\log |P|}{(|P|^{1+2r} - 1)(|P| + 1)}. \tag{8.2.6}$$

Therefore, the ratios conjecture implies that the following holds

**Theorem 8.2.1.** *Assuming Conjecture 8.1.1,  $\frac{1}{\log |D|} \ll \mathfrak{R}(r) < \frac{1}{4}$  and  $\mathfrak{I}(r) \ll_\varepsilon |D|^{1-\varepsilon}$  we have*

$$\begin{aligned} & \sum_{D \in \mathcal{H}_{2g+1, q}} \frac{L'(\frac{1}{2} + r, \chi_D)}{L(\frac{1}{2} + r, \chi_D)} \\ &= \sum_{D \in \mathcal{H}_{2g+1, q}} \left( \frac{\zeta'_A(1+2r)}{\zeta_A(1+2r)} + A'_D(r; r) - (\log q) |D|^{-r} \mathcal{X}(\frac{1}{2} + r) \zeta_A(1-2r) A_D(-r; r) \right) \\ & \quad + O(|D|^{1/2+\varepsilon}), \end{aligned} \tag{8.2.7}$$

where  $A_D(\alpha; \gamma)$  is defined in (8.1.21).

## 8.3 The One–Level Density Formula

Now we are in a position to derive the formula for the one–level density for the zeros of quadratic Dirichlet  $L$ –functions over function fields, complete with lower order terms.

Let  $\gamma_D$  denote the ordinate of a generic zero of  $L(s, \chi_D)$  on the half–line (remember that here, unlike from the number field case, we do not need to assume that all of the complex zeros are on the half–line, because the Riemann hypothesis is established for this family of  $L$ –functions). As  $L(s, \chi_D)$  is a

functions of  $u = q^{-s}$  and so is periodic with period  $2\pi i / \log q$  we can confine our analysis of the zeros for the range  $-\pi i / \log q \leq \Im(s) \leq \pi i / \log q$ . We consider the one-level density

$$S_1(f) := \sum_{D \in \mathcal{H}_{2g+1, q}} \sum_{\gamma_D} f(\gamma_D), \quad (8.3.1)$$

where  $f$  is an even  $(2\pi / \log q)$ -periodic test function and holomorphic.

By Cauchy's theorem we have

$$S_1(f) = \sum_{D \in \mathcal{H}_{2g+1, q}} \frac{1}{2\pi i} \left( \int_{(c)} - \int_{(1-c)} \right) \frac{L'(s, \chi_D)}{L(s, \chi_D)} f(-i(s - 1/2)) ds, \quad (8.3.2)$$

where  $(c)$  denotes a vertical line from  $c - \pi i / \log q$  to  $c + \pi i / \log q$  and  $3/4 > c > 1/2 + 1/\log |D|$ . The integral on the  $c$ -line is

$$\frac{1}{2\pi} \int_{-\pi/\log q}^{\pi/\log q} f(t - i(c - 1/2)) \sum_{D \in \mathcal{H}_{2g+1, q}} \frac{L'(1/2 + (c - 1/2 + it), \chi_D)}{L(1/2 + (c - 1/2 + it), \chi_D)} dt. \quad (8.3.3)$$

The sum over  $D$  can be replaced by Theorem 8.2.1 (see the 1-level density section of [Conr-Sna] for a more detailed analysis). Next we move the path of integration to  $c = 1/2$  as the integrand is regular at  $t = 0$  to obtain

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi/\log q}^{\pi/\log q} f(t) \sum_{D \in \mathcal{H}_{2g+1, q}} \left( \frac{\zeta'_A(1 + 2it)}{\zeta_A(1 + 2it)} + A'_D(it; it) \right. \\ \left. - (\log q) |D|^{it} \mathcal{X}(\tfrac{1}{2} + it) \zeta_A(1 - 2it) A_D(-it; it) \right) dt + O(|D|^{1/2+\epsilon}). \end{aligned} \quad (8.3.4)$$

Now for the integral on the  $(1 - c)$ -line, we make the following variable change, letting  $s \rightarrow 1 - s$ , and we use the functional equation (6.1.13) to write

$$\frac{L'(1 - s, \chi_D)}{L(1 - s, \chi_D)} = \frac{\mathcal{X}'_D(s)}{\mathcal{X}_D(s)} - \frac{L'(s, \chi_D)}{L(s, \chi_D)}, \quad (8.3.5)$$

where

$$\frac{\mathcal{X}'_D(s)}{\mathcal{X}_D(s)} = -\log |D| + \frac{\mathcal{X}'(s)}{\mathcal{X}(s)}. \quad (8.3.6)$$

So, finally, we obtain the following theorem

**Theorem 8.3.1.** *Assuming the Ratios Conjecture 8.1.1, the one–level density for the zeros of the family of quadratic Dirichlet  $L$ –functions associated with hyperelliptic curves given by the affine equation  $C_D : y^2 = D(x)$ , where  $D \in \mathcal{H}_{2g+1,q}$  is given by*

$$\begin{aligned} S_1(f) &= \sum_{D \in \mathcal{H}_{2g+1,q}} \sum_{\gamma_D} f(\gamma_D) \\ &= \frac{1}{2\pi} \int_{-\pi/\log q}^{\pi/\log q} f(t) \sum_{D \in \mathcal{H}_{2g+1,q}} \left( \log |D| + \frac{\mathcal{X}'}{\mathcal{X}}\left(\frac{1}{2} - it\right) + 2 \left( \frac{\zeta'_A(1+2it)}{\zeta_A(1+2it)} \right. \right. \\ &\quad \left. \left. + A'_D(it; it) - (\log q)|D|^{it} \mathcal{X}\left(\frac{1}{2} + it\right) \zeta_A(1-2it) A_D(-it; it) \right) \right) dt \\ &\quad + O(|D|^{1/2+\varepsilon}), \end{aligned} \quad (8.3.7)$$

where  $\gamma_D$  is the ordinate of a generic zero of  $L(s, \chi_D)$  and  $f$  is an even and periodic suitable test function.

### 8.3.1 The Scaled One–Level Density

Defining

$$f(t) = h\left(\frac{t(2g \log q)}{2\pi}\right) \quad (8.3.8)$$

and scaling the variable  $t$  as

$$\tau = \frac{t(2g \log q)}{2\pi}, \quad (8.3.9)$$

we get after a change of variables

$$\begin{aligned}
& \sum_{D \in \mathcal{H}_{2g+1,q}} \sum_{\gamma_D} h\left(\gamma_D \frac{(2g \log q)}{2\pi}\right) \\
&= \frac{1}{2g \log q} \int_{-g}^g h(\tau) \sum_{D \in \mathcal{H}_{2g+1,q}} \left( \log |D| + \frac{\mathcal{X}'}{\mathcal{X}} \left( \frac{1}{2} - \frac{2\pi i \tau}{2g \log q} \right) + 2 \left( \frac{\zeta'_A(1 + \frac{4\pi i \tau}{2g \log q})}{\zeta_A(1 + \frac{4\pi i \tau}{2g \log q})} \right. \right. \\
&+ \mathcal{A}'_D \left( \frac{2\pi i \tau}{2g \log q}; \frac{2\pi i \tau}{2g \log q} \right) - (\log q) e^{-(2\pi i \tau / 2g \log q) \log |D|} \\
&\times \left. \left. \mathcal{X} \left( \frac{1}{2} + \frac{2\pi i \tau}{2g \log q} \right) \zeta_A \left( 1 - \frac{4\pi i \tau}{2g \log q} \right) A_D \left( -\frac{2\pi i \tau}{2g \log q}; \frac{2\pi i \tau}{2g \log q} \right) \right) \right) d\tau + O(|D|^{1/2+\varepsilon}).
\end{aligned} \tag{8.3.10}$$

Writing

$$\zeta_A(1+s) = \frac{1/\log q}{s} + \frac{1}{2} + \frac{1}{12}(\log q)s + O(s^2), \tag{8.3.11}$$

we have

$$\frac{\zeta'_A(1+s)}{\zeta_A(1+s)} = -s^{-1} + \frac{1}{2} \log q - \frac{1}{12}(\log q)^2 s + O(s^3). \tag{8.3.12}$$

For large  $g$  only the  $\log |D|$  term, the  $\zeta'_A/\zeta_A$  term and the final term in the integral contribute, yielding the asymptotic

$$\begin{aligned}
& \sum_{D \in \mathcal{H}_{2g+1,q}} \sum_{\gamma_D} h\left(\gamma_D \frac{(2g \log q)}{2\pi}\right) \\
&\sim \frac{1}{2g \log q} \int_{-\infty}^{\infty} h(\tau) \left( (\#\mathcal{H}_{2g+1,q}) \log |D| \right. \\
&\quad \left. - \frac{(\#\mathcal{H}_{2g+1,q})(2g \log q)}{2\pi i \tau} + (\#\mathcal{H}_{2g+1,q}) \frac{e^{-2\pi i \tau}}{2\pi i \tau} (2g \log q) \right) d\tau.
\end{aligned} \tag{8.3.13}$$

But, since  $h$  is an even function, we can ignore the middle term and the last term can be duplicated with a change of sign of  $\tau$ , leaving

$$\begin{aligned} \lim_{g \rightarrow \infty} \frac{1}{\#\mathcal{H}_{2g+1,q}} \sum_{D \in \mathcal{H}_{2g+1,q}} \sum_{\gamma_D} h \left( \gamma_D \frac{(2g \log q)}{2\pi} \right) \\ = \int_{-\infty}^{\infty} h(\tau) \left( 1 - \frac{\sin(2\pi\tau)}{2\pi\tau} \right) d\tau. \end{aligned} \quad (8.3.14)$$

Thus for  $q$  fixed and in the large  $g$  limit, the one–level density of the scaled zeros has the same form as the one–level density of the eigenvalues of the matrices from  $USp(2g)$  chosen with respect to Haar measure as can be seen from part (ii) Theorem 1.4.2 and so our result is in agreement with results previously obtained by Rudnick [Rud-TPFHE].

And as final conclusion, we can say that the ratios conjecture for the  $L$ –functions in this family confirm a conjecture of Katz and Sarnak, that to leading order of the low–lying zeros for this ensemble have symplectic statistics.

**Remark 8.3.2.** *The calculations presented in this chapter also appears in [And-Kea12].*

**Remark 8.3.3.** *We can compare the formula obtained in the function field case (8.3.10) with the formula (3.14) obtained by Conrey and Snaith [Conr-Sna], and observing that the function  $\mathcal{X}(s)$  plays the role of the  $\Gamma(s)$  in the function field setting we can see the striking resemblance between the two formulas.*

*An interesting question proposed by Rudnick (private communication) is if we can go beyond the leading term and compare our results with the one–level density formula presented in Corollary 2 of [Rud-TPFHE], which is:*

**Corollary 8.3.4** (Rudnick). *If  $f$  is an even test function in the Schwartz space, with Fourier transform  $\hat{f}$  supported in  $(-2, 2)$ , then*

$$\langle Z_f \rangle = \int_{USp(2g)} Z_f(U) dU + \frac{\text{dev}(f)}{g} + o\left(\frac{1}{g}\right), \quad (8.3.15)$$

where

$$Z_f(U) := \sum_{j=1}^N F(\theta_j), \quad \text{with} \quad F(\theta_j) := \sum_{k \in \mathbb{Z}} f \left( N \left( \frac{\theta}{2\pi} - k \right) \right), \quad (8.3.16)$$

where  $e^{i\theta_j}$  are the eigenvalues of the unitary  $N \times N$  matrix  $U$  and

$$\text{dev}(f) = \hat{f}(0) \sum_{\substack{P \text{ monic} \\ \text{irreducible}}} \frac{\deg(P)}{|P|^2 - 1} - \hat{f}(1) \frac{1}{q-1}. \quad (8.3.17)$$

Now if we take the comparison further, we have the term corresponding to  $\int_{USp(2g)} Z_f(U) dU$  is given by the term  $(2g+1)(\log q)$ , which comes from the  $\log |D|$  term in (8.3.10), after the division by  $\#\mathcal{H}_{2g+1,q}$ . Also we have that the corresponding term to the sum over prime polynomials is given by the term  $A'_D$  in (8.3.10) after we perform the division by  $\#\mathcal{H}_{2g+1,q}$ .

But the main question raised by Rudnick is about the term that appears in his result which has  $q$  dependence  $\hat{f}(1) \frac{1}{q-1}$ . Such term at this stage seems mysterious when we look at the equation (8.3.10). A brief look at the equation (8.3.10) shows us that their terms has no  $q$  dependence as the  $q$  dependence in Rudnick's theorem.

Seems interesting to analyze carefully the last term that appears in (8.3.10) because the  $q$  dependence should come of such term. At this point the thesis author is not able to get the  $q$  dependence from (8.3.10) and put all the terms in one-one comparison with the Rudnick's result. So, a research problem for the near future is to identify whether the Ratios Conjecture produces the same result derived by Rudnick, or if the the Ratios Conjecture is not able to present the deviation term.

Also seems a good time for reviewing the results presented by Rudnick, since our formula is in complete agreement presented with the formulae for the number-field case.

# Chapter 9

## Conclusion and Further Questions

The initial motivation for this thesis was to investigate the moments of  $L$ -functions associated with curves over a fixed finite field  $\mathbb{F}_q$  in the limit when the genus of the curve grows, i.e.,  $g \rightarrow \infty$ . The study of similar questions, fixing  $q$  and letting  $g \rightarrow \infty$ , was initiated by Kurlberg & Rudnick [Kur-Rud] and by Faifman & Rudnick [Fai-Rud] and so this thesis is an attempt to enlarge our mathematical knowledge of such questions and therefore the limit considered in this thesis is precisely the opposite of those studied by Katz and Sarnak in [Katz-Sar99a, Katz-Sar99b]. Following the philosophy of Katz and Sarnak, if  $q \rightarrow \infty$  we can use the RMT powerful results, for example, to compute moments of  $L$ -functions, but the same philosophy does not apply when we fix  $q$  and investigate the limit  $g \rightarrow \infty$ , since in this case we do not have the Equidistribution theorems and so the questions about moments become, purely, a number theory question.

The theorem established in Chapter 3 indicates the same techniques developed to study and prove a formula for the first moment of  $L(s, \chi_D)$  also can be extended to study the second and third power moment for this case. And as we are working on the function field setting, we expect that the problem about higher moments are treatable in this setting, based in the old philosophy that

problems in function fields are easier to solve.

In Chapter 5, making use of the same techniques developed in previous chapters, we obtain averages for  $h_D$  which agree with the analogous averages previously obtained in the context of number fields. However it would be interesting to continue study the class number problems over function fields and try to determine, unconditionally, if there are infinitely many real quadratic function fields with class number one. This definitely is a research problem which deserves attention in the future.

Going back to Chapter 6 and Chapter 7 we see that the same heuristic arguments developed by Conrey *et.al.* [CFKRS05, Conr-Far-Zir] can be developed in the function field setting. Thus, a natural question is to extend the same heuristics for unitary and orthogonal families of  $L$ -functions in the function field context, and we expect that the formulae developed will be very similar to those developed in the number field case.

Understand problems about distribution of zeros of  $L$ -functions and moments in the context of function fields is interesting, once we hope to gain a better understanding of similarities and differences between number fields and function fields and we dream that one day we can totally understand these parallel worlds and make a translation of the proof of Riemann hypothesis for curves to the context of number fields.

Below we present some problem ideas that can be approachable in the function field setting:

1. Using the ideas presented in [Conr-Far-Zir], [Conr-Sna] and making use of the Ratio Conjectures for  $L$ -functions over function fields presented in Chapter 7 we hope to obtain the two-level density, three-level density and the  $n$ -level density of zeros of  $L(s, \chi_D)$  in this setting.
2. Using the ideas presented in [Bog-Kea95, Bog-Kea96a, Bog-Kea96b, Bog-Leb, EHM, Pol06, Pol08] we hope to use the analogue of Hardy-Littlewood twin prime conjecture for monic irreducible polynomials to obtain the



pair-correlation of zeros of the  $L$ -functions over function fields. We also hope use the pair-correlation to obtain the Hardy–Littlewood twin prime conjecture for this case.

3. Investigate higher moments for  $L$ -functions associated with hyperelliptic curves when both  $q, g \rightarrow \infty$  and  $q$  is small compared to  $g$ , for example,  $q = \log \log \log g$ .
4. Try to establish asymptotic formulas for the first few moments ( $k = 1$ , and possibly, for  $k = 2$ ) for

$$\sum_{\chi \pmod{Q}}^* |L(\frac{1}{2}, \chi)|^{2k} \tag{9.0.1}$$

as  $\deg(Q) \rightarrow \infty$ , where  $Q \in \mathbb{F}_q[T]$  is a monic polynomial of positive degree and the sum is over all primitive characters  $\chi \pmod{Q}$ .

5. Translate the Li's Criterion [Li] for Riemann Hypothesis to  $L$ -functions over function fields and try to produce a new proof the Riemann Hypothesis for curves using the function field Li's Criterion.
6. Investigate the Mollification techniques in this setting and do the translation of the Levinson's method over function fields.
7. Investigate Log Moments of  $L(s, \chi_D)$ .
8. Beyond the function field context, we intend start to think about moments of the Selberg zeta function, Witten zeta function and Shintani zeta function and at same time find the correct RMT models for these  $L$ -functions.

# Appendix A

## The Leading Order Term for the First Moment Using the Function Field Tauberian Theorem

In this appendix we obtain the leading order term for the first moment of quadratic Dirichlet  $L$ -functions making use of the function field version of the Wiener–Ikehara Tauberian theorem.

From Chapter 3, we have that

$$\sum_{D \in \mathcal{H}_{2g+1,q}} L\left(\frac{1}{2}, \chi_D\right) \sim \frac{P(1)}{2\zeta_A(2)} |D| \log_q |D|, \quad (\text{A.1})$$

as  $|D| \rightarrow \infty$ , i.e.,  $g \rightarrow \infty$ . Recall that  $|D| = q^{2g+1}$  and

$$P(s) = \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left(1 - \frac{1}{(|P| + 1)|P|^s}\right). \quad (\text{A.2})$$

The leading order for the first moment is obtained when we analyze the following expressions

$$\frac{|D|}{\zeta_A(2)} \sum_{\substack{n=0 \\ 2|n}}^g q^{-n/2} \sum_{\substack{l \text{ monic} \\ \deg(l)=n/2}} \prod_{P|l} (1 + |P|^{-1})^{-1}, \quad (\text{A.3})$$

and

$$\frac{|D|}{\zeta_A(2)} \sum_{\substack{m=0 \\ 2|n}}^{g-1} q^{-m/2} \sum_{\substack{l \text{ monic} \\ \deg(l)=m/2}} \prod_{P|l} (1 + |P|^{-1})^{-1}, \quad (\text{A.4})$$

as can be seen from (3.4.35) and from the dual sum presented in Chapter 3.

In this Appendix we obtain the same leading order term, as expected, by using the following theorem quoted from [Rosen, Theorem 17.1]

**Theorem A.0.5.** *Let  $f : \mathcal{D}_K^+ \rightarrow \mathbb{C}$  be given and  $\zeta_f(s)$  be the following absolutely convergent Dirichlet series for  $\Re(s) > 1$*

$$\zeta_f(s) = \sum_{N=0}^{\infty} F(N)q^{-Ns}, \quad (\text{A.5})$$

where  $F(N) = \sum_{\deg(D)=N} f(D)$ . Moreover, let  $\zeta_f(s)$  be holomorphic on  $\{s \in B \mid \Re(s) = 1\}$  with a simple pole at  $s = 1$ , where

$$B = \left\{ s \in \mathbb{C} \mid -\frac{\pi i}{\log(q)} \leq \Im(s) < \frac{\pi i}{\log(q)} \right\}. \quad (\text{A.6})$$

Then, there is a  $\delta < 1$  such that

$$F(N) = \alpha \log(q)q^N + O(q^{\delta N}), \quad (\text{A.7})$$

with  $\alpha = \text{Res}_{s=1} \zeta_f(s)$  and  $\mathcal{D}_K^+$  is the set of monic polynomials in  $A = \mathbb{F}_q[T]$ .

The calculations now presented are in the same spirit of the calculations given in the Appendix section of the recent paper by Keating and Rudnick [Kea-Rud].

Let  $f(l) = \prod_{P|l} (1 + |P|^{-1})^{-1}$  and denote by  $\zeta_f(s)$  the following Dirichlet series,

$$\zeta_f(s) = \sum_{l \text{ monic}} \frac{f(l)}{|l|^s} = \sum_{N=0}^{\infty} \sum_{\substack{l \text{ monic} \\ \deg(l)=N}} f(l)q^{-Ns}. \quad (\text{A.8})$$

We have that,

$$\begin{aligned}
 \zeta_f(s) &= \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left( 1 + f(P) \sum_{j=1}^{\infty} \left( \frac{1}{|P|^s} \right)^j \right) \\
 &= \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left( 1 + \frac{|P|}{|P|+1} \frac{1}{|P|^s-1} \right) \\
 &= \zeta_A(s) \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left( 1 + \frac{|P|}{|P|+1} \frac{1}{|P|^s-1} \right) \left( 1 - \frac{1}{|P|^s} \right). \quad (\text{A.9})
 \end{aligned}$$

Now, applying the Theorem A.0.5 we have that

$$\sum_{\substack{l \text{ monic} \\ \text{deg}(l)=N}} f(l) = \alpha \log(q) q^N + O(q^{\delta N}), \quad (\text{A.10})$$

for some  $\delta < 1$ .

Computing the residue of the simple pole at  $s = 1$  from (A.9) we obtain that,

$$\begin{aligned}
 \alpha &= \frac{1}{\log(q)} \prod_P \left( 1 + \frac{|P|}{(|P|+1)(|P|-1)} \right) \left( 1 - \frac{1}{|P|} \right) \\
 &= \frac{1}{\log(q)} \prod_P \left( 1 - \frac{1}{|P|(|P|+1)} \right). \quad (\text{A.11})
 \end{aligned}$$

Therefore, in the end, we obtain that

$$\sum_{\substack{l \text{ monic} \\ \text{deg}(l)=n/2}} f(l) = P(1) q^{n/2} + O(q^{\frac{\delta n}{2}}). \quad (\text{A.12})$$

Now making the substitution of (A.12) in (A.3) we have that

$$\begin{aligned}
 \frac{|D|}{\zeta_A(2)} \sum_{\substack{n=0 \\ 2|n}}^g q^{-n/2} \sum_{\substack{l \text{ monic} \\ \text{deg}(l)=n/2}} \prod_{P|l} (1 + |P|^{-1})^{-1} \\
 = \frac{|D|}{\zeta_A(2)} P(1) [g/2] + O\left(|D| q^{\frac{\varepsilon g}{2} + \varepsilon}\right), \quad (\text{A.13})
 \end{aligned}$$

where  $\varepsilon = \delta - 1 < 0$ .

Proceeding in the same manner for the dual sum (A.4) we have that,

$$\begin{aligned} \frac{|D|}{\zeta_A(2)} \sum_{\substack{m=0 \\ 2|n}}^{g-1} q^{-m/2} \sum_{\substack{l \text{ monic} \\ \deg(l)=m/2}} \prod_{P|l} (1 + |P|^{-1})^{-1} \\ = \frac{|D|}{\zeta_A(2)} P(1) \left[ \frac{g-1}{2} \right] + O\left(|D|q^{\frac{\varepsilon}{2}g - \frac{\varepsilon}{2}}\right). \end{aligned} \quad (\text{A.14})$$

Putting together the equations (A.13) and (A.14) we obtain precisely the leading order term for the first moment, as desired.

# Appendix B

## Brief Review on Probability

### Theory

Let  $\Omega$  be a topological space, and let  $2^\Omega$  denote the set of all subsets of  $\Omega$ .

**Definition B.0.6.**  $\mathcal{A} \subseteq 2^\Omega$  is a  $\sigma$ -**algebra** of  $\Omega$  if

1.  $\Omega \in \mathcal{A}$
2. If  $A \in \mathcal{A}$  then  $A^C \in \mathcal{A}$ .
3.  $\mathcal{A}$  is closed under countable unions and countable intersections.

**Definition B.0.7.** The **Borel**  $\sigma$ -**algebra** associated with the topological space  $\Omega$  is the  $\sigma$ -algebra generated by the open sets (i.e. the smallest  $\sigma$ -algebra containing all the open sets in that topology).

**Definition B.0.8.** A **probability measure** defined on  $\sigma$ -algebra  $\mathcal{A}$  of  $\Omega$  is a function  $\mathbb{P} : \mathcal{A} \mapsto [0, 1]$  that satisfies

1.  $\mathbb{P}\{\Omega\} = 1$
2. For every countable sequence  $\{A_n\}_{n \geq 1}$ , where  $A_i \in \mathcal{A}$  which are pairwise disjoint,

$$\mathbb{P}\left\{\bigcup_{n=1}^{\infty} A_n\right\} = \sum_{n=1}^{\infty} \mathbb{P}\{A_n\}. \quad (\text{B.1})$$

**Definition B.0.9.** Let  $\mathcal{F}$  be the Borel  $\sigma$ -algebra of a topological space  $F$ . A function  $X : \Omega \mapsto F$  is a **random variable** if  $X^{-1}(\Lambda) \in \mathcal{A}$  for all  $\Lambda \in \mathcal{F}$ .

We use the usual notation and write  $\mathbb{P}\{X \in \Lambda\} = \mathbb{P}\{\omega : X(\omega) \in \Lambda\}$  for the probability that a random variable  $X$  takes a value lying in some set  $\Lambda \in \mathcal{F}$ . For example, if  $F = \mathbb{R}^d$  we have,

**Definition B.0.10.** We say  $X$  has a **probability density function**  $p : \Omega \mapsto \mathbb{R}^d$  if for all  $\Lambda \in \mathcal{F}$

$$\mathbb{P}\{X \in \Lambda\} = \int_{\Lambda} p(x)dx. \quad (\text{B.2})$$

**Definition B.0.11.**  $\mathbb{E}f(X)$  denotes the **expectation** of  $f(X)$ , and if  $X$  has a density function,

$$\mathbb{E}f(X) = \int_{\mathbb{R}^d} f(x)p(x)dx. \quad (\text{B.3})$$

**Definition B.0.12.** The **moment generating function** of  $X$  is  $\mathbb{E}e^{\langle \lambda, X \rangle}$ , and the **characteristic function** of  $X$  is  $\mathbb{E}e^{i\langle \lambda, X \rangle}$ , where  $\langle \lambda, X \rangle = \sum_{j=1}^d \lambda_j X_j$ .

For example, if  $F = \mathbb{R}$  and  $c(\lambda) = \mathbb{E}e^{i\lambda X}$  then by Fourier inversion, if  $p(\cdot)$  exists then

$$p(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda x} c(\lambda) d\lambda. \quad (\text{B.4})$$

**Definition B.0.13.** An **ensemble** is just a set with a probability measure attached to it.

# Appendix C

## The Gamma function and the Barnes $G$ -function

### C.1 Euler's Gamma function

The Euler Gamma function is discussed in [Gra-Ryz] and is defined for  $\Re(z) > 0$  by

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt. \quad (\text{C.1})$$

#### C.1.1 Properties of the Gamma function.

(i) [Integral representation]:

$$\frac{1}{\Gamma(z)} = \frac{i}{2\pi} \int_{\mathcal{C}} (-t)^{-z} e^{-t} dt \quad (\text{C.2})$$

where  $\mathcal{C}$  starts at  $+\infty$  on the real axis, circles the origin once in the counter-clockwise direction, and returns to the starting point.

(ii) [Functional equation]:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \quad (\text{C.3})$$

(iii) [Recurrence relation]:  $\Gamma(z+1) = z\Gamma(z)$ .



(iv) [Complex conjugation]:  $\Gamma^*(z) = \Gamma(z^*)$ .

(v) [Multiplication formula]:

$$\Gamma(z)\Gamma\left(z + \frac{1}{m}\right)\Gamma\left(z + \frac{2}{m}\right)\cdots\Gamma\left(z + \frac{m-1}{m}\right) = (2\pi)^{(m-1)/2}m^{1/2-mz}\Gamma(mz). \quad (\text{C.4})$$

(vi) [Stirling's asymptotic formula]: For  $|z| \rightarrow \infty$  with  $|\arg(z)| < \pi$ ,

$$\begin{aligned} \log \Gamma(z) \sim (z - \tfrac{1}{2}) \log z - z + \tfrac{1}{2} \log(2\pi) + \frac{1}{12z} - \frac{1}{360z^3} + \cdots \\ + \frac{B_{2m}}{2m(2m-1)z^{2m-1}} + \cdots \end{aligned} \quad (\text{C.5})$$

where the  $B_{2m}$  are the Bernoulli numbers, defined by

$$\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \frac{t}{e^t - 1}. \quad (\text{C.6})$$

(vii) [Taylor expansion]: For  $|z| < 2$

$$\log \Gamma(z+1) = -\log(z+1) + z(1-\gamma) + \sum_{n=2}^{\infty} (-1)^n (\zeta(n) - 1) \frac{z^n}{n}, \quad (\text{C.7})$$

where  $\gamma$  is the Euler–Mascheroni constant.

(viii) [Special values and poles]:  $\Gamma(z)$  has simple poles at  $z = -n$ ,  $n = 0, 1, 2, \dots$  of residue  $(-1)^n/n!$ .

$$\Gamma(1) = 1.$$

$$\Gamma(1/2) = \sqrt{\pi}.$$

$$\Gamma(1) = 1 \text{ and } \Gamma(n) = (n-1)! \text{ for positive integers } n.$$

## C.2 The Barnes $G$ -function

The Barnes  $G$ -function is defined [Barnes] as

$$G(z+1) = (2\pi)^{z/2} \exp(-(z(z+1) + \gamma z^2)/2) \prod_{n=1}^{\infty} \left[ \left(1 + \frac{z}{n}\right)^n \exp\left(-z + \frac{z^2}{2n}\right) \right], \quad (\text{C.1})$$

with  $\exp(x) = e^x$ .

### C.2.1 Properties of the Barnes $G$ -function

(i) [Recurrence relation]:  $G(z + 1) = \Gamma(z)G(z)$ .

(ii) [Complex conjugation]:  $G^*(z) = G(z^*)$ .

(iii) [Multiplication formula]:

$$G(nz) = K(n)n^{n^2z^2/2-nz}(2\pi)^{-\frac{n^2-n}{2}z} \prod_{i=0}^{n-1} \prod_{j=0}^{n-1} G\left(z + \frac{i+j}{n}\right), \quad (\text{C.2})$$

where

$$K(n) = e^{-(n^2-1)\zeta'(-1)} n^{\frac{5}{12}} (2\pi)^{(n-1)/2}. \quad (\text{C.3})$$

(iv) [Asymptotic expansion]: For  $|z| \rightarrow \infty$  with  $|\arg(z)| < \pi$ ,

$$\log G(z+1) \sim z^2 \left( \frac{1}{2} \log z - \frac{3}{4} \right) + \frac{1}{2} z \log 2\pi - \frac{1}{12} \log z + \zeta'(-1) + O\left(\frac{1}{z}\right). \quad (\text{C.4})$$

(v) [Taylor expansion]: For  $|z| < 1$ ,

$$\log G(z + 1) = \frac{1}{2}(\log 2\pi - 1)z - \frac{1}{2}(1 + \gamma)z^2 + \sum_{n=3}^{\infty} (-1)^{n-1} \zeta(n-1) \frac{z^n}{n}. \quad (\text{C.5})$$

(vi) [Special values and zeros]:  $G(z + 1)$  has zeros at  $z = -n$  of order  $n$ , where  $n = 1, 2, \dots$

$G(1)=1$ .

$G(1/2)=e^{3\zeta'(-1)/2}\pi^{-1/4}2^{1/24}$ .



# Bibliography

- [Ake-Baik-DiFran] G. Akeman, J. Baik and P. Di Francesco *The Oxford Handbook of Random Matrix Theory*, Oxford University Press, Oxford, 2011.
- [Andrade] J.C. Andrade, *A Note on the Mean Value of  $L$ -functions in Function Fields*, preprint (2012), to appear in the Int. J. Number Theory.
- [And-Kea11] J.C. Andrade and J.P. Keating, *The Mean Value of  $L(\frac{1}{2}, \chi)$  in the Hyperelliptic Ensemble*, preprint (2011), submitted to the J. Number Theory.
- [And-Kea12] J.C. Andrade and J.P. Keating, *Conjectures for the Integral Moments and Ratios of  $L$ -functions over Function Fields*, preprint (2012).
- [Apostol] T.M. Apostol, *Introduction to Analytic Number Theory*, Springer-Verlag, New York, 1976.
- [Artin] E. Artin, *Quadratische Körper in Geibiet der Höheren Kongruenzen I and II*. Math. Z., **19**(1924), 153–296.
- [Bal-Conr-HB] R. Balasubramanian, J.B. Conrey and D.R. Heath-Brown, *Asymptotic mean square of the product of the Riemann zeta-function and a Dirichlet polynomial*, J. Reine Angew. Math. **357** (1985), 161–181.
- [Bal-Ram] R. Balasubramanian, K. Ramachandra, *Proof of some conjectures on the mean-value of Titchmarsh series I*, Hardy-Ramanujan J. **13** (1990), 1–20.
- [Barnes] E.W. Barnes, *The theory of  $G$ -function*, J. Lond. Math. Soc., **31** (1900), 264–314.
- [Ber-Gel] J. Bernstein and S. Gelbart, *An Introduction to the Langlands Program*, Birkhäuser, Boston, 2004.
- [Ber-Kea] M.V. Berry and J.P. Keating,  *$H = xp$  and the Riemann zeros*, in Supersymmetry and Trace Formulae: Chaos and Disorder, eds. I.V.

- 
- Lerner, J.P. Keating and D.E. Khmelnitskii (Plenum Press), (1999), 355–367.
- [Bog-Kea95] E.B. Bogomolny and J.P. Keating, *Random matrix theory and the riemann zeros I: three- and four-point correlations*, Nonlinearity **8** (1995), 1115–1131.
- [Bog-Kea96a] E.B. Bogomolny and J.P. Keating, *Gutzwiller’s trace formula and spectral statistics: Beyond the diagonal approximation*, Phys. Rev. Lett., **77** (1996), 1472–1475.
- [Bog-Kea96b] E.B. Bogomolny and J.P. Keating, *Random matrix theory and the riemann zeros II:  $n$ -point correlations*, Nonlinearity **9** (1996), 911–935.
- [Bog-Leb] E.B. Bogomolny and P. Leboeuf, *Statistical properties of the zeros of zeta functions – beyond the riemann case*, Nonlinearity **7**(4) (1997), 1155–1167.
- [Boh-Lan] H. Bohr and E. Landau, *Sur les zeros de la fonctions  $\zeta(s)$  de Riemann*, C. R. Acad. des Sciences Paris **158** (1914), 158–162.
- [BDFL] A. Bucur, C. David, B. Feigon and M. Lalin, *Statistics for traces of cyclic trigonal curves over finite fields*, Int. Math. Res. Not. IMRN no.5 (2010), 932–967.
- [Buc-Diac] A. Bucur and A. Diaconu, *Moments of quadratic Dirichlet  $L$ -functions over rational function fields*, Mosc. Math. J. **10** (2010), no. 3, 485–517.
- [Bui-Conr-Young] H.M. Bui, J.B. Conrey and M.P. Young, *More than 41% of the zeros of the zeta function are on the critical line*, preprint 2011 arXiv:1002.4127v2.
- [Bui-HB] H.M. Bui and D.R. Heath–Brown, *A note on the fourth moment of Dirichlet  $L$ -functions*, Acta Arith. **141** (2010), 335–344.
- [Carlitz] L. Carlitz, *The arithmetic of polynomials in a finite field*, Amer. J. Math. **54** (1932), 476–492.
- [Car-Jaf-Wil] J. Carlson, A. Jaffe and A. Wiles, *The Millennium Prize Problems*, AMS CMI, (2006).
- [Conr-MTF] J.B. Conrey, *More than two fifths of zeros of the Riemann zeta function are on the critical line*, J. Reine Angew. Math. **399** (1989), 1–26.
- [Conr-NFPMR] J.B. Conrey, *A note on the fourth power moment of the Riemann zeta-function*, Analytic Number Theory, Progress in Mathematics **138** (1996).

- [Conr-Far] J.B. Conrey and D.W. Farmer, *Mean values of  $L$ -functions and symmetry*, Int. Math. Res. Not. IMRN **17**(2000), 883–908.
- [CFKRS03] J.B. Conrey, D.W. Farmer, J.P. Keating, M.O. Rubinstein and N.C. Snaith, *Autocorrelation of Random Matrix Polynomials*, Commun. Math. Phys., **237**(2003), 365–395.
- [CFKRS05] J.B. Conrey, D.W. Farmer, J.P. Keating, M.O. Rubinstein and N.C. Snaith, *Integral moments of  $L$ -functions*, Proc. Lond. Math. Soc. **91**(2005), 33–104.
- [CFKRS08] J.B. Conrey, D.W. Farmer, J.P. Keating, M.O. Rubinstein and N.C. Snaith, *Lower order terms in the full moment conjecture for the Riemann zeta function*, J. Number Theory, **128**(2008), 1516–1554.
- [Conr-Far-Zir] J.B. Conrey, D.W. Farmer and M.R. Zirnbauer, *Autocorrelation of ratios of  $L$ -functions*, Commun. Number Theory Phys., vol. 2, **3** (2008), 593–636.
- [Conr-For-Sna] J.B. Conrey, P. Forrester and N.C. Snaith, *Averages of ratios of characteristic polynomials for the classical compact groups*, Int. Math. Res. Not. no. 7 (2005), 397–431.
- [Conr-Gho84] J.B. Conrey and A. Ghosh, *On mean values of the zeta-function*, Mathematika **31** (1984), 159–161.
- [Conr-Gho92] J.B. Conrey and A. Ghosh, *Mean values of the Riemann zeta-function III*, Proceedings of the Amalfi Conference on Analytic Number Theory, Università di Salerno, (1992).
- [Conr-Gho98] J.B. Conrey and A. Ghosh, *A conjecture for the sixth power moment of the Riemann zeta-function*, Int. Math. Res. Not. IMRN **15** (1998), 775–780.
- [Conr-Gon] J.B. Conrey and S.M. Gonek, *High moments of the Riemann zeta-function*, Duke Math. J. **107**, no. 3 (2001), 577–604.
- [Conr-Iwan-Sound] J.B. Conrey, H. Iwaniec and K. Soundararajan, *The sixth power moment of Dirichlet  $L$ -functions*, (2012), arXiv:0710.5176v2.
- [Conr-Sna] J.B. Conrey and N.C. Snaith, *Applications of the  $L$ -Functions Ratios Conjectures*, Proc. Lond. Math. Soc. **94** (2007), 594–646.
- [Conr-Sound] J.B. Conrey and K. Soundararajan, *Real zeros of quadratic Dirichlet  $L$ -functions*, Invent. Math. **150** No.1, (2002) 1–44.
- [Daven] H. Davenport, *Multiplicative Number Theory*, 3rd edition, Springer-Verlag, Graduate Texts in Mathematics **74**, New York, 2000.

- 
- [Deligne I] P. Deligne, *La conjecture de Weil I*, Inst. Hautes Études Sci. Publ. Math. **48** (1974), 273–308.
- [Deligne II] P. Deligne, *La conjecture de Weil II*, Inst. Hautes Études Sci. Publ. Math. **52** (1981), 313–428.
- [Diac-Gold-Hoff] A. Diaconu, D. Goldfeld and J. Hoffstein, *Multiple Dirichlet series and moments of zeta and L-functions*, Compos. Math. **139** (2003), 297–360.
- [Dyson] F.J. Dyson, *Statistical theory of energy levels of complex systems, I, II and III*, J. Math. Phys. **3** (1962), 140–156, 157–165, 166–175.
- [Dyson72] F.J. Dyson, *Missed opportunities*, Bull. Amer. Math. Soc. Volume 78, Number **5** (1972), 635–652.
- [Edwards] H.M. Edwards, *Riemann’s zeta function*, Academic Press 1974.
- [Eff-Hay] G.W. Effinger and D.R. Hayes, *Additive Number Theory of Polynomials Over a Finite Field*, Clarendon Press, Oxford University Press, New York, 1991.
- [EHM] G.W. Effinger, K.H. Hicks and G.L. Mullen, *Twin irreducible polynomials over finite fields*, Finite Fields with Applications in Coding Theory, Cryptography and Related Areas, (edited by G. L. Mullen, H. Stichtenoth and H. Tapia-Recillas), Springer, 2002, 94–111.
- [Euler] L. Euler, *Variae observationes circa series infinitas*, Comm. Acad. Sci. Petropolitanae **9** (1737), 222–236 (also in “Opera” (1), Vol 14, 216–244).
- [Faifman] D. Faifman, *Counting zeros of L-functions over the rational function field*, MSc. Thesis, Tel-Aviv, 2008.
- [Fai-Rud] D. Faifman and Z. Rudnick, *Statistics of the zeros of zeta functions in families of hyperelliptic curves over a finite field*, Compos. Math. **146**(2010), 81–101.
- [Far-LMR] D.W. Farmer, *Long mollifiers of the Riemann zeta-function*, Mathematika **40** (1993), 71–87.
- [For-Sna-Ver] P.J. Forrester, N.C. Snaith and J.J.M. Verbaarschot *Developments in random matrix theory*, J. Phys. A: Math. Gen., **36**:R1–R10, 2003.
- [Gao05] P. Gao, *n-level density of the low-lying zeros of quadratic Dirichlet L-functions*, PhD Thesis (2005), University of Michigan.
- [Gao08] P. Gao, *n-level density of the low-lying zeros of quadratic Dirichlet L-functions*, preprint (2008), arXiv:0806.4830v1.

- [Gauss] C.F. Gauss, *Letter to Encke*, dated 24 December (1849), Werke, Vol. II, 444–447.
- [Gauss-DA] C.F. Gauss, *Disquisitiones Arithmeticae*, Engl. transl., (New Haven–London, 1966).
- [Gel] S. Gelbart, *An Elementary Introduction to the Langlands Program*, Bull. Amer. Math. Soc. **10** (1984), 177–219.
- [Gold-Vio] D. Goldfeld, C. Viola, *Mean values of  $L$ -functions associated to elliptic, Fermat and other curves at the centre of the critical strip*, J. Number Theory **11** (1979), 305–320.
- [Goss] D. Goss, *Basic Structures of Function Field Arithmetic*, Springer-Verlag, New York, 1998.
- [Gra-Ryz] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products*, Seventh Edition, Academic Press, Oxford, 2007.
- [Hadamard] J. Hadamard, *Sur la distribution des zeros de la fonction  $\zeta(s)$  et ses consequences arithmétiques*, Bull. Soc. Math. France **24** (1896), 199–220.
- [Har-SZR] G.H. Hardy, *Sur les zéros de la fonction  $\zeta(s)$  de Riemann*, C.R. Acad. Sci. Paris **158** (1914), 1012–1014.
- [Har-Lit] G.H. Hardy and J.E. Littlewood, *Contributions to the theory of the Riemann zeta-function and the theory of the distribution of primes*, Acta Math. **41** (1918), 119–196.
- [Har-Lit23] G.H. Hardy and J.E. Littlewood, *Some problems in “Partitio Numerorum” III: On the expression of a number as a sum of primes*, Acta Math. **44** (1923), 1–70.
- [HB-FMRZ] D.R. Heath-Brown, *The fourth power moment of the Riemann zeta function*, Proc. Lond. Math. Soc. **38** (1979), 385–422.
- [HB-FMR] D.R. Heath-Brown, *Fractional moments of the Riemann zeta function*, J. Lond. Math. Soc. **24** (1981), 65–78.
- [HB-FMDL] D.R. Heath-Brown, *The fourth power mean of Dirichlet’s  $L$ -functions*, Analysis 1 (1981), 25–32.
- [HB-FMRII] D.R. Heath-Brown, *Fractional moments of the Riemann zeta function II*, Q. J. Math. **44** (1993), 185–197.
- [HB-Hux] D.R. Heath-Brown and M.N. Huxley, *Exponential sums with a difference*, Proc. Lond. Math. Soc. **61** (1990), 227–250.



- 
- [Hejhal] D.A. Hejhal, *On the triple correlation of zeros of the zeta function*, Int. Math. Res. Not., **7**(1994), 293–302.
- [Hoff-Ros] J. Hoffstein and M. Rosen, *Average values of  $L$ -series in function fields*, J. Reine Angew. Math. **426**(1992), 117–150.
- [Huc-Put-Zir] A. Huckleberry, A. Püttmann and M.R. Zirnbauer *Haar expectations of ratios of random characteristic polynomials*, preprint ArXiv:0709:1215.
- [Hug-Rud] C.P. Hughes and Z. Rudnick, *Linear statistics and low-lying zeros of  $L$ -functions*, Q. J. Math., **54**(3) (2003), 309–333.
- [Hux-MHZ] M.N. Huxley, *The large sieve inequality for algebraic number fields II. Means of moments of Hecke zeta-functions*, Proc. Lond. Math. Soc. **21** (1970), 108–128.
- [Huy-Kea-Sna] D.K. Huynh, J.P. Keating and N.C. Snaith, *Lower order terms for the one-level density of elliptic curve  $L$ -functions*, J. Number Theory, Volume **129**, (2009), 2883–2902.
- [Ing] A.E. Ingham, *Mean-value theorems in the theory of the Riemann zeta-function*, Proc. Lond. Math. Soc. **27** (1926), 273–300.
- [Ire-Ros] K. Ireland and M. Rosen, *A Classical Introduction to Modern Number Theory*, Springer-Verlag, GTM Vol. 84, New York, 1990.
- [Ivic] A. Ivić, *The Riemann zeta-function: Theory and applications*, Dover Publications, 2003.
- [Iwan-CEC] H. Iwaniec, *Conversations on the exceptional character*, Analytic Number Theory, Lecture Notes in Mathematics, Springer (2006), 97–130.
- [Iwan-Kow] H. Iwaniec and E. Kowalski, *Analytic Number Theory*, Amer. Math. Soc. Colloq. Publ. **53**, Amer. Math. Soc., Providence, 2004.
- [Iwan-Luo-Sar] H. Iwaniec, W. Luo and P. Sarnak, *Low lying zeros of families of  $L$ -functions*, Publications Mathématiques de l’IHES, **91** (2000), 55–131.
- [Iwan-Sar] H. Iwaniec and P. Sarnak, *Dirichlet  $L$ -functions at the central point*, Number theory in progress, Vol. 2, de Gruyter, Berlin (1999), 941–952.
- [Jutila] M. Jutila, *On the mean value of  $L(\frac{1}{2}, \chi)$  for real characters*, Analysis 1 (1981), 149–161.

- [Katz-Sar99a] N.M. Katz and P. Sarnak, *Random Matrices, Frobenius eigenvalues, and monodromy*, American Mathematical Society Colloquium Publications, vol. 45, American Mathematical Society, Providence, RI, 1999.
- [Katz-Sar99b] N.M. Katz and P. Sarnak, *Zeroes of zeta functions and symmetry*, Bull. Amer. Math. Soc. (N.S.) **36** (1999), 1–26.
- [Kea93] J.P. Keating, *The riemann zeta function and quantum chaology*, In Quantum Chaos; editors, G. Casati, I. Guarneri, and U. Smilansky Amsterdam (1993), North Holland, 145–185.
- [Kea-Lin-Rud] J.P. Keating, N. Linden and Z. Rudnick, *Random matrix theory, the exceptional Lie groups, and L-functions*, J. Phys. A, **36** (2003), 2933–2944.
- [Kea-Odg] J.P.Keating and B.E. Odgers, *Symmetry Transitions in Random Matrix Theory and L-functions*, Comm. Math. Phys. **281** (2008), 499–528.
- [Kea-Rud] J.P. Keating and Z. Rudnick, *The variance of the number of prime polynomials in short intervals and in residue classes*, preprint (2012), arXiv:1204.0708v2.
- [Kea-Sna00a] J.P. Keating and N.C. Snaith, *Random matrix theory and  $\zeta(\frac{1}{2} + it)$* , Comm. Math. Phys. **214** (2000), 57–89.
- [Kea-Sna00b] J.P. Keating and N.C. Snaith, *Random matrix theory and L-functions at  $s = \frac{1}{2}$* , Comm. Math. Phys. **214** (2000), 91–110.
- [Kim-Sar] H. Kim, *Functorality for the exterior square of  $GL_4$  and the symmetric fourth of  $GL_2$* . With appendix 1 by D.Ramakrishnan and appendix 2 by Kim and Sarnak, J. Amer. Math. Soc. **16** (2003), 139–183.
- [Kur-Rud] P. Kurlberg and Z. Rudnick, *The fluctuations in the number of points on a hyperelliptic curve over a finite field*, J. Number Theory, **129**(2009), 580–587.
- [Lau] A. Laurinćikas, *Limit theorems for the Riemann zeta function*, Kluwer Academic Publishers, 1996.
- [Legendre] A.M. Legendre, *Essai sur la Theorie des Nombres*, Paris: Duprat, (1798).
- [Levinson] N. Levinson, *More than one-third of the zeros of Riemann’s zeta-function are on  $\sigma = 1/2$* , Adv. Math. **13** (1974), 383–436.
- [Li] X. Li, *The positivity of a sequence of numbers and the Riemann hypothesis*, J. Number Theory **65**, (2)(1997), 325–333.

- 
- [Lipschitz] R. Lipschitz, Sitzungber. Akad. Berlin (1865), 174–185.
- [Lun-Rie-Win] J. van der Lune, H.J.J. te Riele, D.T. Winter, *On the zeros of the Riemann zeta function in the critical strip IV*, Math. Comp. **46** (1986), 667–681.
- [Mehta] M.L. Mehta, *Random matrices*, Academic Press, 3rd edition, 2004.
- [Mez-Sna] F. Mezzadri and N.C. Snaith, *Recent perspectives in random matrix theory and number theory*, London Mathematical Society Lecture Note Series 322, London Mathematical Society, 2005.
- [Mon-TMNT] H.L. Montgomery, *Topics in multiplicative number theory*, Springer-Verlag, Berlin (1971).
- [Mon-PCZR] H. L. Montgomery, *The pair correlation of zeros of the zeta function*, Analytic Number Theory, Proc. Sym. Pure Math. **24** (1973), 181–193.
- [Mon-Vau73] H.L. Montgomery and R.C. Vaughan, *The large sieve*, Mathematika, **20**(1973), 119–134.
- [Mon-Vau74] H.L. Montgomery and R.C. Vaughan, *Hilbert's inequality*, J. Lond. Math. Soc., **8**(1974), 73–82.
- [Mon-Vau] H.L. Montgomery and R.C. Vaughan, *Multiplicative number theory I - Classical theory*, Cambridge studies in advanced Mathematics 97, Cambridge University Press, 2007.
- [Mot-NMVL II] Y. Motohashi, *A note on the mean value of the zeta function and L-functions II*, Proc. Japan Acad. **61**(1985), 313–316.
- [Mot-NMVL III] Y. Motohashi, *A note on the mean value of the zeta function and L-functions III*, Proc. Japan Acad. **62**(1985), 152–154.
- [Odl89] A.M. Odlyzko, *The  $10^{20}$ -th zero of the Riemann zeta function and 70 million of its neighbours*, Preprint 1989.
- [Odl] A.M. Odlyzko, *The  $10^{20}$ -th zero of the Riemann zeta function and 175 million of its neighbors*, ATT Bell Laboratories preprint, <http://www.dtc.umn.edu/odlyzko/>.
- [Ozl-Sny93] A.E. Özlük and C. Snyder, *Small zeros of quadratic L-functions*, Bull. Austral. Math. Soc., **47**(1993), 307–319.
- [Ozl-Sny99] A.E. Özlük and C. Snyder, *On the distribution of the nontrivial zeros of quadratic L-functions close to the real axis*, Acta Arith., **91**(1999), 209–228.

- [Paley] R.E.A.C. Paley, *On the  $k$ -analogues of some theorems in the theory of the Riemann zeta-function*, Proc. Lond. Math. Soc. (2) **32** (1931), 273–311.
- [Pol06] P. Pollack, *An explicit approach to Hypothesis H for polynomials over finite fields*, Anatomy of integers. Proceedings of a conference on the anatomy of integers, Montreal, March 13th-17th, 2006 eds: J.M. de Koninck, A. Granville and F. Luca, pp. 259–273.
- [Pol08] P. Pollack, *A polynomial analogue of the twin prime conjecture*, Proc. Amer. Math. Soc. (11) **136** (2008), 3775–3784.
- [Poussin] C.J. de la Vallée Poussin, *Recherches analytiques la théorie des nombres premiers*, Ann. Soc. scient. Bruxelles **20**, No. 2 (1896), 183–256.
- [Rad-Sound] M. Radziwill and K. Soundararajan, *Continuous Lower Bounds for Moments of Zeta and  $L$ -functions*, preprint (2012), arXiv:1202.1351v1.
- [Ram-SRII] K. Ramachandra, *Some remarks on the mean value of the Riemann zeta-function and other Dirichlet series II*, Hardy-Ramanujan J. **3** (1980), 1–25.
- [Ram-SRIII] K. Ramachandra, *Some remarks on the mean value of the Riemann zeta-function and other Dirichlet series III*, Hardy-Ramanujan J. **5** (1980), 145–180.
- [Riemann] G.F.B. Riemann, *Über die Anzahl der Primzahlen unter einer gegebenen Grösse*, (On the number of primes less than a given quantity), Monatsber. Knigl. Preuss. Akad. Wiss. Berlin (1859), 671–680.
- [Rosen] M. Rosen, *Number Theory in Function Fields*, Graduate Texts in Mathematics, vol. 210, Springer-Verlag, New York, 2002.
- [Rub-ESIZ] M.O. Rubinstein, *Evidence for a spectral interpretation of the zeros of  $L$ -functions*. Ph.D. thesis, Princeton University, June 1998.
- [Rub-LZR] M.O. Rubinstein, *Low lying zeros of  $L$ -functions and random matrix theory*, Duke Math. J., **109** (2001), no. 1, 147–181.
- [Rud-TPFHE] Z. Rudnick, *Traces of high powers of the Frobenius class in the hyperelliptic ensemble*, Acta Arith., **143**(2010), 81–99.
- [Rud-Sar] Z. Rudnick and P. Sarnak, *Zeros of principal  $L$ -functions and random matrix theory*, Duke Math. J., **81** (1996), 269–322.
- [Rud-Sound05] Z. Rudnick and K. Soundararajan, *Lower bounds for moments of  $L$ -functions*, Proc. Natl. Acad. Sci. USA **102** (2005), 6837–6838.

- 
- [Rud-Sound06] Z. Rudnick and K. Soundararajan, *Lower bounds for moments of  $L$ -functions: symplectic and orthogonal examples*, Multiple Dirichlet Series, Automorphic Forms, and Analytic Number Theory, Proc. Symp. Pure Math. AMS 75 (2006).
- [Sel-ZR] A. Selberg, *On the zeros of Riemann's zeta-function*, Skr. Norske Vid. Akad. Oslo. **10** (1942), 1–59.
- [Siegel] C.L. Siegel, *The average measure of quadratic forms with given determinant and signature*, Ann. Math. **45** (1944), 667–685.
- [Sound-MVRZ] K. Soundararajan, *Mean values of the Riemann zeta-function*, Mathematika **42** (1995), 158–174.
- [Sound-NQDL] K. Soundararajan, *Nonvanishing of quadratic Dirichlet  $L$ -functions at  $s = \frac{1}{2}$* , Ann. Math. **152** (2000), 447–488.
- [Sound-FMDL] K. Soundararajan, *The fourth moment of Dirichlet  $L$ -functions*, Clay Mathematics Proceedings, Vol. 7, (2007) 239–242.
- [Sound-MRZ] K. Soundararajan, *Moments of the Riemann zeta-function*, Ann. Math., **170** (2009), no. 2, 981–993.
- [Thakur] D.S. Thakur, *Function Field Arithmetic*, World Scientific Publishing Co., NJ, 2004.
- [Tit] E.C. Titchmarsh, *The Theory of the Riemann Zeta-function*. Second edition. Edited and with a preface by D. R. Heath-Brown. The Clarendon Press, Oxford University Press, New York, 1986.
- [Tra-Wid] C.A. Tracy and H. Widom, *Introduction to random matrices*, Geometric and Quantum Aspects of Integrable Systems (Ed. G.F. Helminck), Springer–Verlag Lecture Notes in Physics **424**(1993), 103–130.
- [Ulmer] D. Ulmer, *Function Fields and Random Matrices*, in “Ranks of elliptic curves and random matrix theory”, London Mathematical Society Lecture Note Series **341** (2007), 109–142.
- [VM] H. von Mangoldt, *Zur Verteilung der Nullstellen der Riemannschen Funktionen  $\xi(t)$* , Math. Ann. **60** (1905) 1–19.
- [Watt] N. Watt, *A note on the mean square of  $|\zeta(\frac{1}{2} + it)|$* , J. Lond. Math. Soc. **82** (2) (2012), 279–294.
- [Weil-CAD] A. Weil, *Sur les Courbes Algébriques et les Variétés qui s'en Déduisent* (Hermann, Paris, 1948).
- [Weil-CA] A. Weil, *Sur les Courbes Algébriques et variétés abéliennes*, Hermann, Paris, 1971.

- [Weil-BNT] A. Weil, *Basic Number Theory*, Springer–Verlag, New York, 1973.
- [Weyl-CG] H. Weyl, *Classical groups*, Princeton University Press, 1946.
- [Wigner] E. Wigner, *Characteristic vectors of bordered matrices with infinite dimensions*, Ann. Math. **62** (3) (1955), 548–564.
- [Young-FMQDL] M. Young, *The first moment of quadratic Dirichlet  $L$ -functions*, Acta Arith. **138** (2009), no. 1, 73–99.
- [Young-FMDL] M. Young, *The fourth moment of Dirichlet  $L$ -functions*, Ann. Math. **173** (2011), 1–50.
- [Young-TMQDL] M. Young, *The third moment of quadratic Dirichlet  $L$ -functions*, preprint (2012), arXiv:1203.4457v1.