ANALYTIC NUMBER THEORY IN FUNCTION FIELDS - TCC 2015
PROBLEM SHEET 1

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1-) If $m \in A = \mathbb{F}_q[T]$, and $\deg(m) > 0$, show that $q - 1 \mid \Phi(m)$.

2-) If $q = p$ is a prime number and $P \in A$ is an irreducible, show $(\mathbb{F}_q[T]/P^2A)^*$ is cyclic if and only if $\deg(P) = 1$.

3-) Suppose $m \in A$ is monic and that $m = m_1 m_2$ is a factorization into two monics which are relatively prime and of positive degree. Show $(A/mA)^*$ is not cyclic except possibly in the case $q = 2$ and m_1 and m_2 have relatively prime degrees.

4-) Assume $q \neq 2$. Determine all m for which $(A/mA)^*$ is cyclic (see the proof of Proposition 1.6 in [1]).

5-) Suppose $d \mid q - 1$. Show $x^d \equiv -1 (\mathrm{mod}\ P)$ is solvable if and only if $(-1)^{\frac{q-1}{d}\deg(P)} = 1$.

6-) Show $\prod_{\alpha \in \mathbb{F}_q^*} \alpha = -1$.

7-) Let $P \in A$ be a monic irreducible. Show

$$\prod_{\substack{\deg(f) < d \\ f \text{ monic}}} f \equiv \pm 1 (\mathrm{mod}\ P),$$

where $d = \deg(P)$. Determine the sign on the right-hand side of this congruence.

8-) For an integer $m \geq 1$ define $[m] = T^{q^m} - T$. Show that $[m]$ is the product of all monic irreducible polynomials $P(T)$ such that $\deg(P)$ divides m.

9-) Working in the polynomial ring $\mathbb{F}_q[u_0, u_1, \ldots, u_n]$, define $D(u_0, \ldots, u_n) = \det |u_i^{q^j}|$, where $i, j = 0, 1, \ldots, n$. This is called the Moore determinant.

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Show

$$D(u_0, u_1, \ldots, u_n) = \prod_{i=0}^{n} \prod_{c_{i-1} \in \mathbb{F}_q} \cdots \prod_{c_0 \in \mathbb{F}_q} (u_i + c_{i-1}u_{i-1} + \cdots + c_0 u_0).$$

Hint: Show each factor on the right divides the determinant and then count degrees.

10-) Define $F_j = \prod_{i=0}^{j-1}(T^{q^j} - T^{q^i}) = \prod_{i=0}^{j-1}[j-i]^{q^i}$. Show that

$$D(1, T, T^2, \ldots, T^n) = \prod_{j=0}^{n} F_j.$$

Hint: Use the fact that $D(1, T, T^2, \ldots, T^n)$ can be viewed as a Vandermonde determinant.

11-) Show that F_j is the product of all monic polynomials in A of degree j.

12-) Define $L_j = \prod_{i=1}^{j}(T^{q^i} - T) = \prod_{i=1}^{j}[i]$. Use Exercise 8 to prove that L_j is the least common multiple of all monics of degree j.

13-) Show

$$\prod_{\deg(f)<d} (u+f) = \frac{D(1, T, T^2, \ldots, T^{d-1}, u)}{D(1, T, T^2, \ldots, T^{d-1})}.$$

14-) Deduce from Exercise 13 that

$$\prod_{\deg(f)<d} (u+f) = \sum_{j=0}^{d} (-1)^{d-j} \frac{F_d}{F_j L_{d-j}^{q^j}} u^{q^j}.$$

15-) Show that the product of all the non-zero polynomials of degree less than d is equal to $(-1)^d F_d / L_d$.

16-) Prove that

$$u \prod_{\substack{\deg(f)<d \\ f \neq 0}} \left(1 - \frac{u}{f}\right) = \sum_{j=0}^{d} (-1)^j \frac{L_d}{F_j L_{d-j}^{q^j}} u^{q^j}.$$

References

[1] M. Rosen: Number Theory in Function Fields, Graduate Texts in Mathematics 210, Springer-Verlag, New York (2002).

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