ANALYTIC NUMBER THEORY IN FUNCTION FIELDS - TCC 2015
PROBLEM SHEET 2

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1-) Let $f \in A$ be a polynomial of degree at least $m \geq 1$. For each $N \geq m$ show that the number of polynomials of degree N divisible by f divided by the number of polynomials of degree N is just $|f|^{-1}$. Thus, it makes sense to say that the probability that an arbitrary polynomial is divisible by f is $|f|^{-1}$.

2-) Let $P_1, P_2, \ldots, P_t \in A$ be distinct monic irreducibles. Give a probabilistic argument that the probability that a polynomial not be divisible by any P_i^2 for $i = 1, 2, \ldots, t$ is given by $\prod_{i=1}^{t}(1 - |P_i|^{-2})$.

3-) Based on Exercise 2, give a heuristic argument to show that the probability that a polynomial in A is square-free is given by $\zeta_A(2)^{-1}$.

4-) Generalize Exercise 3 to give a heuristic to show that the probability that a polynomial in A be k-th power free is given by $\zeta_A(k)^{-1}$.

5-) Show $\sum |m|^{-1}$ diverges, where the sum is over all monic polynomials $m \in A$.

6-) Use the fact that every monic m can be written uniquely in the form $m = m_0 m_1^2$ where m_0 and m_1 are monic and m_0 is square-free to show $\sum |m_0|^{-1}$ diverges where the sum is over all square-free monics m_0.

7-) Use Exercise 6 to show

$$\prod_{\substack{P \text{ irreducible} \\ \deg(P) \leq d}} (1 + |P|^{-1}) \to \infty \qquad \text{as } d \to \infty.$$

8-) Use the obvious inequality $1 + x \leq e^x$ and Exercise 7 to show $\sum |P|^{-1}$ diverges where the sum is over all monic irreducibles $P \in A$.

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9-) Use the Prime Number Theorem for Polynomials to give another proof that $\sum |P|^{-1}$ diverges.

10-) Suppose there were only finitely many monic irreducibles in A. Denote them by $\{P_1, P_2, \ldots, P_n\}$. Let $m = P_1 P_2 \ldots P_n$ be their product. Show $\Phi(m) = 1$ and derive a contradiction.

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