ANALYTIC NUMBER THEORY IN FUNCTION FIELDS
TCC 2015
PROBLEM SHEET 5

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1-) Let $S = \{P_1, P_2, \ldots\}$ be the set of monic primes in A. Let $S_i = \{P_i\}$ be the set consisting of P_i alone. Then, $S \cup_{i=1}^{\infty} S_i$. Show that this implies that Dirichlet density is not countably additive.

2-) Let $P(T) \in A$ and define $N(U^2 = P(T))$ to be the number of pairs $(\alpha, \beta) \in \mathbb{F}_q \times \mathbb{F}_q$ such that $\beta^2 = P(\alpha)$. Show that

$$N(U^2 = P(T)) = \sum_{\alpha \in \mathbb{F}_q} (1 + P(\alpha)^{\frac{q-1}{2}}).$$

3-) Suppose q is odd and let $P \in A$ is a moni irreducible of degree two and that $\chi(a) = (a/P)_2$ for all $a \in A$. Show that $L_A(s, \chi) = 1 \pm q^{-s}$.
(Hint: Use the Reciprocity Law and Exercise 2)

4-) In general, suppose $P \in A$ is a monic irreducible of positive degree and set $\chi(a) = (a/P)_2$. Show that

$$\sum_{\substack{a \text{ monic} \\ \deg(a)=1}} \chi(a) = \pm(q - N(U^2 = P(T))).$$

5-) With the same notation as in Exercise 4, consider the coefficient q^{-s} in $L(s, \chi)$. Use Exercise 4 and the Riemann Hypothesis for function fields to prove

$$|N(U^2 = P(T)) - q| \leq (\deg(P) - 1)\sqrt{q}.$$

6-) Let $h(T) \in A$ be a polynomial of degree m with a non-zero constant term. Show that there are infinitely many primes in A whose first $m + 1$ terms coincide with $h(T)$. What is the Dirichlet density of this collection of primes?

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7-) Let $\{\alpha_1, \alpha_2, \ldots, \alpha_q\}$ be the elements of $\mathbb{F}_q$ labeled in some order and choose elements $\beta_i \mathbb{F}_q^*$ for $i = 1, \ldots, q$, where repetition is allowed. Prove that there are infinitely many primes, $P(T)$, such that $P(\alpha_i) = \beta_i$ for $i = 1, \ldots, q$. What is the Dirichlet density of this set of primes?

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