ANALYTIC NUMBER THEORY IN FUNCTION FIELDS
TCC 2015
PROBLEM SHEET 6

JULIO ANDRADE

1-) Suppose K/F has genus zero. For a divisor D with $\deg D \geq -1$ show that $l(D) = \deg D + 1$.

2-) Suppose K/F has genus zero and that C is a divisor in the canonical class. Show $l(-C) = 3$ and conclude that there is a prime P of degree less than or equal to 2.

3-) Suppose K/F has genus zero and that there is a prime P of degree 1. Show $K = F(x)$ for some element $x \in K$.

4-) Suppose K/F has genus zero and that P is a prime of degree 2. By Exercise 1, $l(P) = 3$. Let $\{1, x, y\}$ be a basis for $L(P)$. Show $K = F(x, y)$. Show further that $\left\{1, x, y, x^2, y^2, xy\right\} \subset L(2P)$ and conclude that x and y satisfy a polynomial of degree 2 over F.

5-) Suppose that K/F has genus 1. Show that $l(D) = \deg D$ for all divisors D with $\deg D \geq 1$.

6-) Suppose K/F has genus 1 and that P is a prime of degree 1. By the last exercise we know $l(2P) = 2$ and $l(3P) = 3$. Let $\{1, x\}$ be a basis of $L(2P)$ and $\{1, x, y\}$ be a basis of $L(3P)$. Show that $K = F(x, y)$. Show also that x and y satisfy a cubic polynomial with coefficients in F of the form

$$Y^2 + a_1XY + a_3Y = X^3 + a_2X^2 + a_4X + a_6.$$

Hint: Consider $L(6P)$.

7-) Let K/F be of positive genus and suppose there is a prime P of degree 1. Suppose further that $L(2P)$ has dimension 2. Let $\{1, x\}$ be a basis. If $\mathrm{char} F \neq 2$, show that there is an element $y \in K$ such that $K = F(x, y)$ and such that x and y satisfy a polynomial equation of the form $Y^2 = f(X)$ where $f(X)$ is a square-free polynomial of degree at least three.

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8-) Use the Riemann-Roch theorem to show that if B and D are divisors such that $B+D$ is in the canonical class, then $|l(B)-l(D)| \leq \frac{1}{2}|\deg(B)-\deg(D)|$.

9-) Suppose P is a prime of degree 1 of a function field K/F. For every positive integer n show $l((n+1)P) - l(nP) \leq 1$.

10-) Let K/F be a function field of genus $g \geq 2$, and P a prime of degree 1. For all integers k we have $l(kP) \leq l((k+1)P)$. If we restrict k to the range $0 \leq k \leq 2g-2$ show there are exactly g values of k where $l(kP) = l((k+1)P)$. These are called Weierstrass gaps. Assume F has characteristic zero. If all the gaps are less than or equal to g we say P is a non-Weierstrass point, if not we say P is a Weierstrass point. It can be shown that there are only finitely many Weierstrass points. In characteristic p there is a theory of Weierstrass points (due to H. Schmid), but the definition is somewhat different.

11-) Suppose K/F has genus 1 and that P_∞ is a prime of degree 1, also called a rational point. Let $E(F)$ denote the set of rational points. If $P, Q \in E(F)$, show there is a unique element $R \in E(F)$ such that $P + Q \sim R + P_\infty$. (Recall that for two divisors A and B, $A \sim B$ means that $A - B$ is a principal divisor). Denote R by $P \oplus Q$. Show that $(P, Q) \to P \oplus Q$ makes $E(F)$ into an abelian group with P_∞ as the zero element.

12-) With the same assumptions as Exercise 11, map $E(F) \to Cl_K^0$ by sending P to the class of $P - P_\infty$. Show that this map is an isomorphism of abelian groups.

13-) Let K/F be a function field and σ an automorphism of K, which leaves F fixed. If $(\mathcal{O}, P)$ is a prime of K, show that $(\sigma\mathcal{O}, \sigma P)$ is also a prime of K. Show, further, that for all $a \in K$, we have $\text{ord}_{\sigma P}(a) = \text{ord}_P(\sigma^{-1}a)$.

14-) (Continuation). The map $P \to \sigma P$ on primes extends to an action of σ on divisors. If $a \in K^*$, show that $\sigma(a) = (\sigma a)$.

15-) (Continuation). If D is a divisor of K, show $a \to \sigma a$ induces a linear isomorphism from $L(D) \to L(\sigma D)$. In particular, if σ fixes D, i.e., $\sigma D = D$, then σ induces an automorphism of $L(D)$.

16-) (Continuation). Suppose P is a prime of degree 1 and that $\sigma P = P$. Then, σ induces an automorphism of $L((2g+1)P)$. If this induced map is the identity, show that σ is the identity automorphism.
Hint: Find two elements $x, y \in K^*$ fixed by σ such that $K = F(x, y)$.

17-) Let A be a divisor and P a prime divisor. Suppose $g \in L(A+P) - L(A)$. If $f \in L(A+P)$ show $f/g \in \mathcal{O}_P$. Use this to prove $l(A+P) \leq l(A) + \deg(P)$.

18-) Use Exercise 17 to show $l(A) \leq \deg(A) + 1$ if A is an effective divisor. Show further that this inequality holds in general. Thus, $l(A)$ is finite for any divisor A.

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