ANALYTIC NUMBER THEORY IN FUNCTION FIELDS
TCC 2015
PROBLEM SHEET 7

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1-) Suppose $f : \mathcal{D}_K^+ \to \mathbb{C}$ and that $\mathrm{Ave}(f)$ exists in the sense defined in Lecture 4. Show that

$$\mathrm{Ave}(f) = \lim_{N\to\infty} \frac{\sum_{\deg D \leq N} f(D)}{\sum_{\deg D \leq N} 1}.$$

2-) Let $D = \sum a(P)P$ be an effective divisor of K. Let $m > 0$ be a positive integer. We say that D is m-th power free if for each P, $a(P) \neq 0$ implies $m \nmid a(P)$. Let f_m be the characteristic function of the m-th power free divisors. Show $\mathrm{Ave}(f_m) = \zeta_K(m)^{-1}$.

3-) Let $m > 0$ be a positive integer and D an effective divisor. Define

$$\sigma_m(D) = \sum_{0 \leq C \leq D} NC^m.$$

Find an asymptotic formula for $S_m(N) = \sum_{\deg D = N} \sigma_m(D)$.

4-) Let $D = \sum a(P)P$ be an effective divisor of K. Define $\mu(D) = 1$ if D is the zero divisor, $\mu(D) = (-1)^t$ if D is square-free and exactly t of the coefficients $a(P)$ are not zero, and $\mu(D) = 0$ otherwise. For every fixed $\varepsilon > 0$, show

$$\sum_{\deg D = N} \mu(D) = O\left(q^{(\frac{1}{2}+\varepsilon)N}\right).$$

5-) Let D be an effective divisor of K. Define $d_m(D)$ to be the number of $m+1$-tuples of effective divisors $(C_1, C_2, \ldots, C_m, C_{m+1})$ such that $\sum_{i=1}^{m+1} C_i = D$. Note that $d_1(D) = d(D)$ is equal to the number of divisors of D. Show that $\zeta_{d_m}(s) = \zeta_K(s)^{m+1}$ and use the theorem from the last lecture to derive an asymptotic formula for $\Delta_m(N) = \sum_{\deg D = N} d_m(D)$.

6-) Using the Selberg's sieve for $\mathbb{F}_q[x]$ prove that $\sum 1/|P|$ converges, where the summation is over all monic irreducible polynomials P such that $P + K$

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is also irreducible.

7-) (Large Sieve for Function Fields). If $\mathcal{A} = \{A_1, \ldots, A_Z\}$, $\deg A_i < 2n$, $Z(P,K)$ is the number of elements of $\mathcal{A}$ which are $\equiv K \pmod{P}$, and $m \leq n$, then

$$\sum_{\deg P \leq m} |P| \sum_{\deg K = \deg P} \left(Z(P,K) - \frac{Z}{|P|} \right)^2 \leq Z q^{2n}.$$

(this is a difficult question)

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