Thm 2.8: There are infinitely many primes p with $p \equiv 3 \bmod 4$.

If $p \equiv 3 \bmod 4$ then $p = 4n+3$ for some $n \in \mathbb{Z}$.

So Thm 2.8 is equivalent to:

Thm 2.8*: There are infinitely many primes of the form $4n+3$.

Proof: To prove this we need first the following two claims:

Claim 1:
If N is odd then N is of the form $4n+1$ or $4n+3$.

PF: N odd, then $N = 2K+1$ for some $K \in \mathbb{Z}$.

- If K is even, then $K = 2n_1$ and therefore $N = 2 \cdot (2n_1)+1 = 4n_1+1$ for some $n_1 \in \mathbb{Z}$.
- If K is odd, then $K = 2n_2+1$ and therefore $N = 2(2n_2+1)+1 = 4n_2+3$ for some $n_2 \in \mathbb{Z}$

Claim 2: Let a and b be two integers of the form $4n+1$. Then $a \cdot b$ is also of the same form.

PF: $a = 4l_1+1$, $b = 4l_2+1$

$$\begin{aligned} a \cdot b &= (4l_1+1)(4l_2+1) \\ &= 16 l_1 l_2 + 4l_1 + 4l_2 + 1 \\ &= 4(4l_1 l_2 + l_1 + l_2) + 1 \\ &= 4K+1 \quad \text{where} \quad K = 4l_1 l_2 + l_1 + l_2. \end{aligned}$$

Proof of Thm 2.8 : (by contradiction)

Suppose there are only finitely many primes of the form $4n+3$, say, $p_0, p_1, \cdots, p_K$, where $p_0 = 3$. Consider the positive integer $N = 4p_1 \cdots p_K + 3$. Clearly, $N > p_K$ and is also of the same form.

case 1: If N itself is a prime, then N would be larger than the largest prime p_K of the form $4n+3$, which is a contradiction.

ase 2: Suppose N is composite. Since N is odd and using Claim 1, we have that every factor of N is of the form $4n+1$ or $4n+3$. If every factor is of the form $4n+1$, then, by Claim 2, N would be of the same form. But, since N is of the form $4n+3$, at least one of the prime factors, say, p, must be of the form $4n+3$.

subcase 1: Let $p = p_0 = 3$. Then $3 \mid N$, so $3 \mid (N-3)$ by Prop 1.4 (iii), that is $3 \mid 4p_1 p_2 \cdots p_K$. So, by Thm 1.23 and Rmk 1.24, i.e., Euclid's Lemma, $3 \mid 2$ or $3 \mid p_i$, where $1 \le i \le K$, but both are impossible.

subcase 2: Let $p = p_i$, where $1 \le i \le K$. Then $p \mid N$ (Factor of N) and $p \mid 4p_1 \cdots p_K$, so $p \mid (N - 4p_1 \cdots p_K)$, that is, $p \mid 3$, again a contradiction.

Both cases lead us to a contradiction, so our assumption must be false. Thus, there is an infinite number of primes of the given form. ∎